A SOUND IN A BOX WITH DIFFERENT COMPLEX IMPEDANCES ON ITS WALLS

Asta Kärkkäinen and Leo Kärkkäinen

Nokia Research Center
Itämerenkatu 11-13, Helsinki, Finland
asta.karkkainen@nokia.com

ABSTRACT

We calculate the pressure in a box with different complex impedances on its walls. The frequency response of a point source is calculated using both Boundary Element Method and analytical calculations combined with Newton iteration to calculate the modes.

1. INTRODUCTION

The sound field in a rectangular box is often used as a simple example system that can be solved analytically in various ways - for example, summing mirror images or by summing the eigenmodes. When the walls have complex impedance, the mirror sources have to be modified accordingly. However, the mirror source cannot take into account the edge diffraction of the box resulting from different impedance on different walls [1]. We have numerically solved the complex eigenfrequency equation that allows us to solve for the pressure in the box. We verify the analytic solution with BEM.

2. THEORY

The pressure field $\boldsymbol{\Psi}$ inside a cavity is the solution for Helmholtz equation [2]

$$\left(\nabla^2 + k^2\right)\Psi = h \tag{1}$$

with boundary conditions

$$\frac{\partial \Psi}{\partial \bar{n}} = -ik\frac{\Psi}{\varsigma}, \tag{2}$$

where k is wave number; $\boldsymbol{\Psi}$ is the pressure field, the harmonic time dependence of which is separated; h is the source term , $\bar{n}$ is the normal of the surface, ς is the impedance. The eigenfunctions $\psi_{n,\ldots,m}(\mathbf{r})$ are solutions of the homogenous Helmholtz equation with the proper boundary conditions and corresponding values of the wave number $k_{n,\ldots,m}$.

$$\nabla^2 \psi_{n,\ldots,m}(\mathbf{r}) + k_{n,\ldots,m}{}^2 \psi_{n,\ldots,m}(\mathbf{r}) = 0\,. \tag{3}$$

Let $\psi(\mathbf{r})$ be the solution for a point source

$$\nabla^2 \psi(\mathbf{r},\mathbf{r}_0) + k^2 \psi(\mathbf{r},\mathbf{r}_0) = Q\delta(\mathbf{r}-\mathbf{r}_0) \tag{4}$$

and express it as a series of all eigenfunctions

$$\psi(\mathbf{r},\mathbf{r}_0) = \sum_{n,\ldots,m} A_{n,\ldots,m}(\mathbf{r}_0)\psi_{n,\ldots,m}(\mathbf{r}) \tag{5}$$

Inserting this to the equation (4) for point source leads to a form

$$\sum_{n,\dots,m} A_{n,\dots,m}(\mathbf{r}_0)\left[\nabla^2\psi_{n,\dots,m}(\mathbf{r}) + k^2\psi_{n,\dots,m}(\mathbf{r})\right] = Q\delta(\mathbf{r}-\mathbf{r}_0)\,. \tag{6}$$

Applying the homogenous equation (3), multiplying by an eigenfunction and finally integrating over the box volume, the equation of the coefficient $A_{n',\dots,m'}(\mathbf{r}_0)$ of the mode becomes

$$\sum_{n,\dots,m} A_{n,\dots,m}(\mathbf{r}_0)\left[-k_{n,\dots,m}{}^2\psi_{n,\dots,m}(\mathbf{r}) + k^2\psi_{n,\dots,m}(\mathbf{r})\right] = Q\delta(\mathbf{r}-\mathbf{r}_0) \Leftrightarrow$$

$$\sum_{n,\dots,m} A_{n,\dots,m}(\mathbf{r}_0)(k^2 - k_{n,\dots,m}{}^2)\psi_{n,\dots,m}(\mathbf{r})\psi_{n',\dots,m'}(\mathbf{r}) = Q\delta(\mathbf{r}-\mathbf{r}_0)\psi_{n',\dots,m'}(\mathbf{r}) \Leftrightarrow$$

$$\sum_{n,\dots,m} A_{n,\dots,m}(\mathbf{r}_0)(k^2 - k_{n,\dots,m}{}^2)\int_\Omega \psi_{n,\dots,m}(\mathbf{r})\psi_{n',\dots,m'}(\mathbf{r})d\Omega = Q\int_\Omega \delta(\mathbf{r}-\mathbf{r}_0)\psi^*{}_{n',\dots,m'}(\mathbf{r})d\Omega \Leftrightarrow \tag{7}$$

$$A_{n',\dots,m'}(\mathbf{r}_0)(k^2 - k_{n',\dots,m'}{}^2)\left[I_{n'x}\times\dots\times I_{m'z}\right] = Q\int_\Omega \delta(\mathbf{r}-\mathbf{r}_0)\psi_{n',\dots,m'}(\mathbf{r})d\Omega = Q\psi^*{}_{n',\dots,m'}(\mathbf{r}_0) \Leftrightarrow$$

$$A_{n',\dots,m'}(\mathbf{r}_0) = Q\frac{\psi_{n',\dots,m'}(\mathbf{r}_0)}{(k^2 - k_{n',\dots,m'}{}^2)\left[I_{n'x}\times\dots\times I_{m'z}\right]}$$

where we have used the orthogonality property of eigenfunctions.

$$\int_\Omega \psi_{n,\dots,m}(\mathbf{r})\psi_{n',\dots,m'}(\mathbf{r})d\Omega = \begin{cases} I_{nx} & \text{if } n,\dots,m = n',\dots,m' \\ 0 & \text{if } n,\dots,m \neq n',\dots,m \end{cases} \tag{8}$$

Hence the response for a point source excitation is

$$\psi(\mathbf{r},\mathbf{r}_0) = Q\sum_{n,\dots,m}\frac{\psi_{n,\dots,m}(\mathbf{r}_0)\psi_{n,\dots,m}(\mathbf{r})}{(k^2 - k_{n,\dots,m}{}^2)\left[I_{nx}\times\dots\times I_{mz}\right]} \tag{9}$$

This applies generally to all shapes of cavities and rooms. Next we will study the eigenfunctions and eigenvalues for the special case of a rectangular room

3. BOX

The system under consideration is a box with locally reacting walls and a point source in it. The complex value of the specific impedance for the absorbing material may be different on each wall. The volume of the box is chosen to be the same as of a car. However the area of it is smaller, since it lacks the details of the car interior that increase the area. The interior domain of the box is bounded by the walls:

$$\begin{cases} 0 < x < l_x \\ 0 < y < l_y\,, \\ 0 < z < l_z \end{cases} \tag{10}$$

where l_x, l_y and l_z are side lengths of the box. Inside the box we have a point source.

The system is solved in cartesian coordinates so that the pressure field is defined as a linear superposition of the products of the solutions of the homogenous Helmholtz equation in three dimensions. The boundary condition determines the eigenvalue equation, which corresponds to the modes.

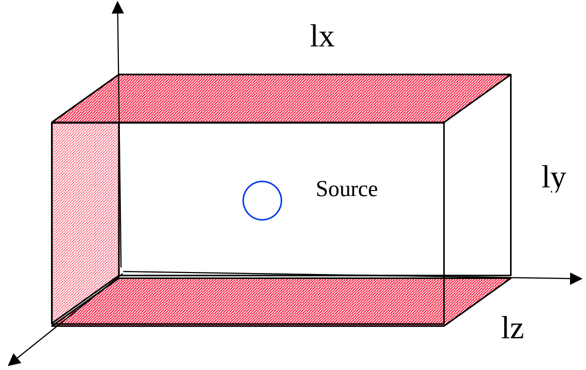

Figure 1. *Box definition.*

3.1. The eigenvalue equation

The box has admittances $\beta_{x=0}, \beta_{x=L\,x}, \beta_{y=0}, \ldots, \beta_{z=L_z}$ The solution of the general Helmholtz equation is a superposition of the eigenfunctions, which are determined by the homogenous Helmholtz equation,

$$\psi(x,\ldots,z) = \sum_{n,\ldots,m} (A_{nx}\cos(k_{nx}x) + B_{nx}\sin(k_{nx}x)) \times \ldots \times (A_{mz}\cos(k_{mz}z) + B_{mz}\sin(k_{mz}z)) \tag{11}$$

The admittances define the boundary conditions,

$$\begin{aligned}
&\left.\frac{\partial\psi(x,y,z)}{\partial x}\right|_{x=0} = -ik\beta_{x=0}\psi(0,y,z) \\
&\left.\frac{\partial\psi(x,y,z)}{\partial x}\right|_{x=L_x} = ik\beta_{x=L_x}\psi(L_x,y,z) \\
&\ldots \\
&\left.\frac{\partial\psi(x,y,z)}{\partial z}\right|_{x=L_z} = ik\beta_{x=L_z}\psi(x,y,L_z)
\end{aligned} \tag{12}$$

which are made true for each component of the sum separately

$$\begin{aligned}
&k_{nx}B_{nx} = -ik\beta_{x=0}A_{nx} \\
&k_{nx}(-A_{nx}\sin(k_{nx}L_x) + B_{nx}\cos(k_{nx}L_x)) = ik\beta_{x=L_x}(A_{nx}\cos(k_{nx}L_x) + B_{nx}\sin(k_{nx}L_x)) \\
&\ldots. \\
&k_{mz}(-A_{mz}\sin(k_{mz}L_z) + B_{mz}\sin(k_{mz}z)) = ik\beta_{x=L_x}(A_{mz}\cos(k_{mz}L_z) + B_{mz}\sin(k_{mz}L_z))
\end{aligned} \tag{13}$$

These are 6 homogenous equations for six unknowns $A_x, B_x, \ldots B_z$. A non-trivial solution is only found if the corresponding 6×6 matrix determinant is zero.

$$\begin{vmatrix}
ik\beta_{x=0} & k_{nx} & 0 & \cdots \\
-ik\beta_{x=L_x}\cos(k_{nx}L_x) - k_{nx}\sin(k_{nx}L_x) & k_{nx}\cos(k_{nx}L_x) - ik\beta_{x=L_x}\sin(k_{nx}L_x) & 0 & \cdots \\
0 & 0 & \ddots & \vdots \\
\vdots & \vdots & \cdots & \vdots
\end{vmatrix} \tag{14}$$

Equation (14) quantizates the values of $k_{nx}, \ldots, k_{mz}$ i.e. it gives the eigenfrequencies $\omega_{n,\ldots,m}$ of the system through equation

$$\left(\frac{\omega_{n,\ldots,m}}{c}\right)^2 = k_{n,\ldots,m}{}^2 \equiv k_{nx}{}^2 + \ldots + k_{mz}{}^2 . \tag{15}$$

The determinant (14) reduces to the product of three 2×2-determinants i.e. the eigenvalues of the 3 coordinate directions are independent. The quantization condition for x-direction is

$$\begin{vmatrix} ik\beta_{x=0} & k_{nx} \\ -ik\beta_{x=L_x}\cos(k_{nx}L_x)-k_{nx}\sin(k_{nx}L_x) & k_{nx}\cos(k_{nx}L_x)-ik\beta_{x=L_x}\sin(k_{nx}L_x) \end{vmatrix} = 0\,, \tag{16}$$

which leads to the transcendental equation

$$\left(\frac{kL_x}{w}\beta_{x=0}\beta_{x=L_x}+\frac{w}{kL_x}\right)\tan(w)+i(\beta_{x=L_x}+\beta_{x=0})=0\,, \tag{17}$$

where $w=kL_x$.Thus the eigensolution for the solution becomes

$$\psi_{n...m}(x,y,z)=\left[\frac{k_{nx}}{k}\cos(k_{nx}x)-i\beta_{x=0}\sin(k_{nx}x)\right]\times...\times\left[\frac{k_{mz}}{k}\cos(k_{mz}z)-i\beta_{z=0}\sin(k_{mz}z)\right]. \tag{18}$$

3.2. Solving the eigenvalue equation

The eigenvalue equation (17),

$$(k_{nx}{}^2+k^2\beta_1\beta_2)\tan(k_{nx}L_x)+ikk_{nx}(\beta_1+\beta_2)=0 \tag{19}$$

cannot be solved analytically. We have used Newton iteration to find the eigenvalues. The algorithm had to be designed in a way that as it finds the solutions they have to remain associated with their proper ordinal number: the n'th zero has to remain the n'th zero. If $k=0$ the equation is easy to solve

$$z\tan(z)=0 \Leftrightarrow \sin(z)=0 \Leftrightarrow z=n\pi\,. \tag{20}$$

We take this as the starting point for the iteration for any frequency. This does not work for the first zero. It is handled by adding a very small number to the starting value. A relaxation parameter α_n is introduced so that it grows linearially from 0 (corresponding to zero frequency) to 1 (corresponding to the target frequency) as the iteration proceeds.

The modified Newton iteration is defined for $f(q_n)=0$ as:

$$q_n{}^{(0)}=n\pi\,,$$

$$q_n{}^{(k+1)}=q_n{}^{(k)}-\alpha_n\frac{f(q_n{}^{(k)})}{f'(q_n{}^{(k)})}\,, \tag{21}$$

When the function f of equation (19) is inserted we are lead to the equation:

$$q_n{}^{(k+1)}=q_n{}^{(k)}-\alpha_n\cos(q_n{}^{(k)})\frac{\left(q_n{}^{(k)}+\dfrac{\beta_2\beta_1k^2L_x{}^2}{q_n{}^{(k)}}\right)\sin(q_n{}^{(k)})+ikL_x(\beta_1+\beta_2)\cos(q_n{}^{(k)})}{\left(1-\beta_2\beta_1\left(\dfrac{kL_x}{q_n{}^{(k)}}\right)^2\right)\sin(q_n{}^{(k)})\cos(q_n{}^{(k)})+\left(q_n{}^{(k)}+\dfrac{\beta_2\beta_1k^2L_x{}^2}{q_n{}^{(k)}}\right)}\,. \tag{22}$$

Without the relaxation parameter α_n the iteration fails to converge to the correct nth root. The algorithm was implemented as a matlab function. Figure 2 shows eigenvalues for frequency range from 20Hz up to 1500Hz.

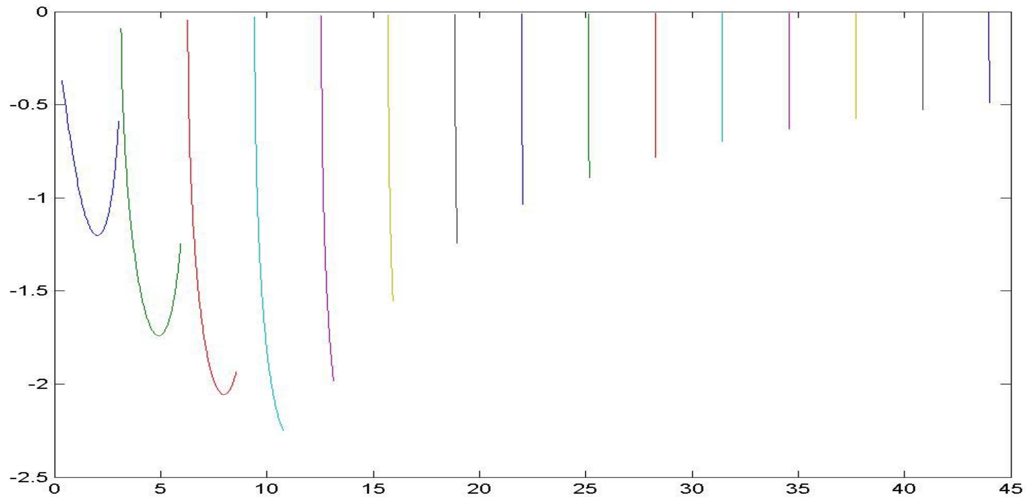

Figure 2 .*Eigenvalues: real (horizontal) and imaginary (vertical) part of* q *. At low frequency the root has a small negative imaginary part. At first it grows with frequency, until at a point it reaches a maximum value corresponding to maximum damping of the mode. As frequency is increased further the imaginary part starts decreasing. At infinite frequency the roots return to the real axis. The real part is increasing all the time, although very little as we get to high frequencies.*

3.3. The eigenfunctions and their normalization

The normalization integral,

$$I_{nx} = \int_0^{L_x} \psi_n(x)\psi_n(x)dx\,, \tag{23}$$

is

$$I_{nx} = \int_0^{L_x} \left[\frac{k_{nx}}{k}\cos(k_{nx}x) - i\beta_{x=0}\sin(k_{nx}x)\right]\left[\frac{k_{nx}}{k}\cos(k_{nx}x) - i\beta_{x=0}\sin(k_{nx}x)\right]dx \tag{24}$$

The integral is calculated with exponential representation of trigonometric functions and using the boundary condition (19). The final form is

$$I_{nx} = \frac{1}{2k^2}\left[L\left(k_n^2 - \beta_{x=0}{}^2k^2\right) - i\,\frac{(\beta_{x=0} + \beta_{x=L})k(k_n^2 - \beta_{x=0}\beta_{x=L}k^2)}{k_n^2 - \beta_{x=L}^2k^2}\right]. \tag{25}$$

Figure 3 and 4 show the real and imaginary parts of the eigenfunctions in tow cases. Figure 3 describes a symmetric case, where the admittances on the opposite walls are the same. Figure 4 shows eigenfunction in a case of different admittances.

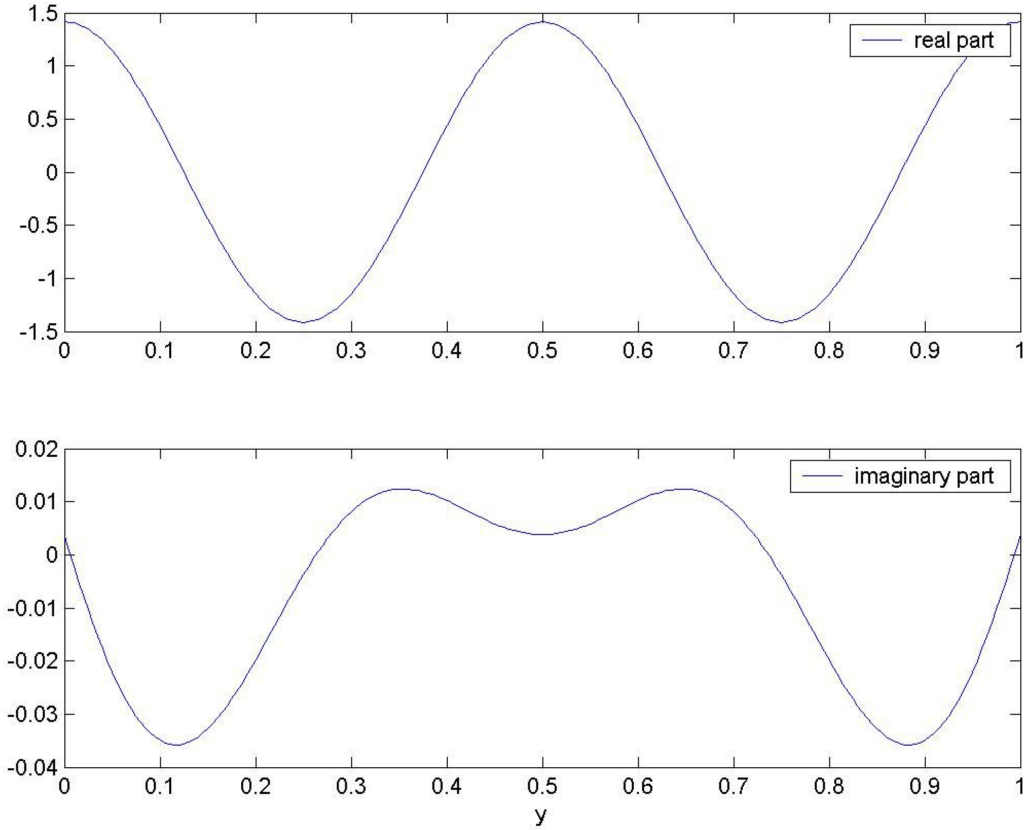

Figure 3 .*The 5th eigenfunction in one direction in case of two absorbing walls with the same admittance. The upper figure describes the real part of the wave function as a function of position. The lower figure corresponds to the imaginary part.*

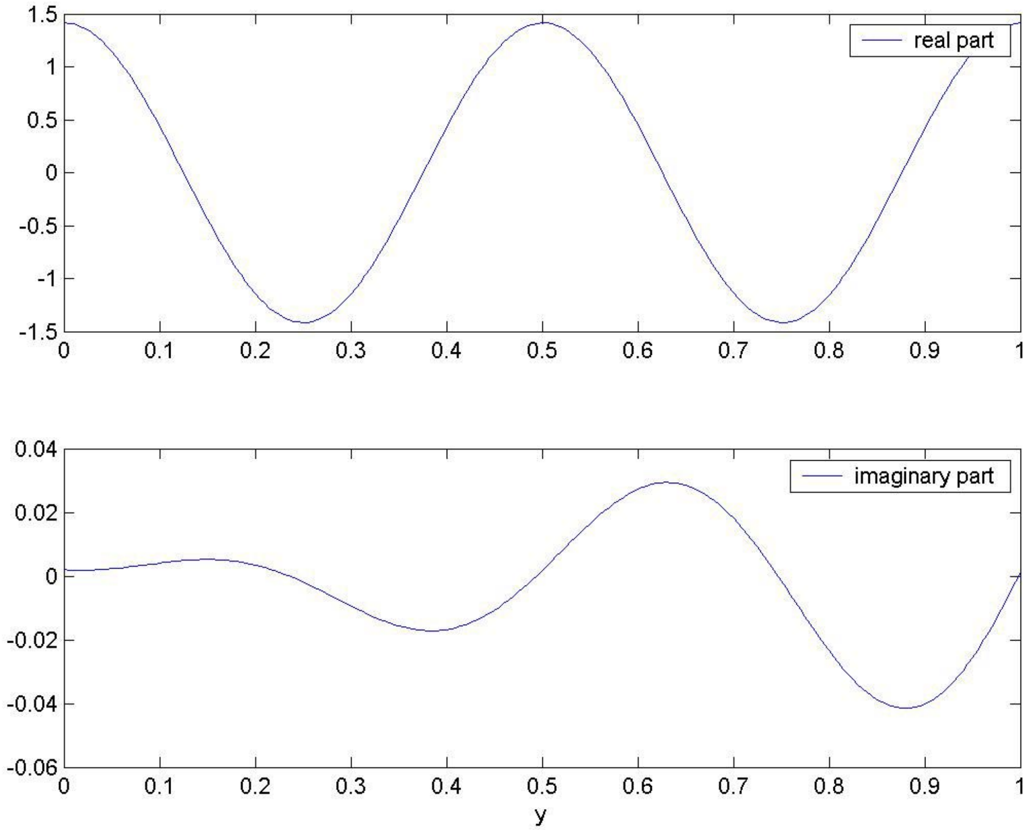

Figure 4 .The 5th eigenfunction in one direction in case of two absorbing walls with different admittances (β=0.01 and (β=038-0.2i) . The upper figure describes the real part of the wave function as a function of position. The lower figure corresponds to the imaginary part.

4. COMPARISON WITH BEM

For BEM model we had a boundary element mesh with 4906 elements. The average element size was 0.05m. In middle of the box we had a point source, the strength of which was 0.005Pa. The boundaries were defined by specific impedance, which was chosen to get real value of $\zeta = 2.6$ at three walls (floor, ceiling and back wall); the other walls were assumed to be rigid.

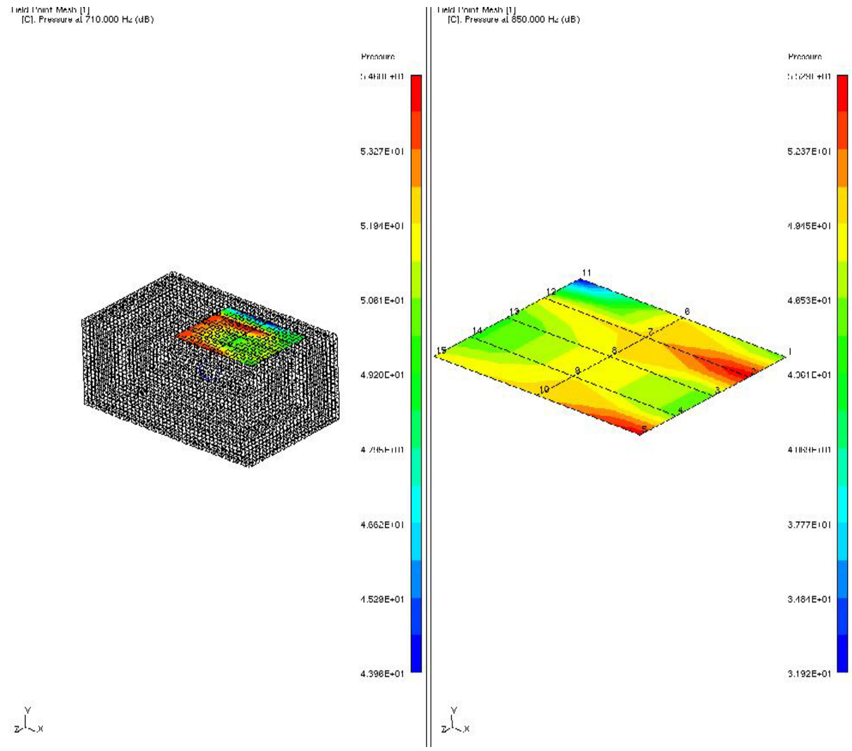

Figure5. *Boundary Element mesh and the plane of field points. SPL (dB) for frequency f=850Hz..*

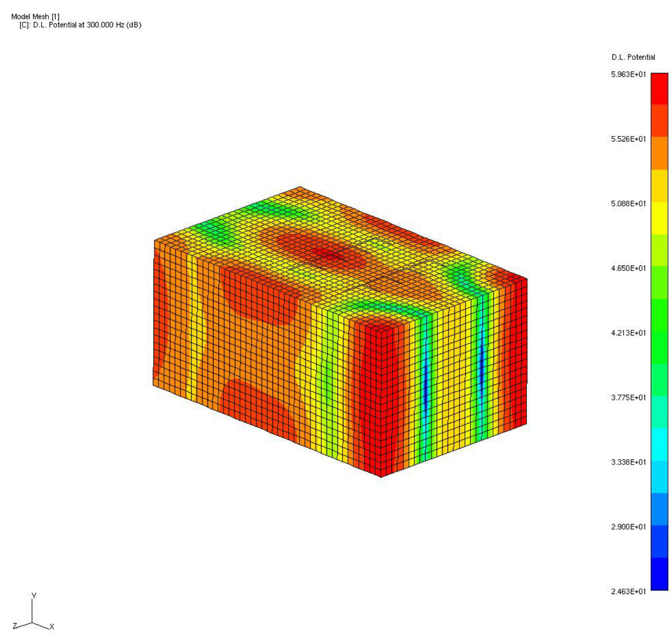

Figure 6: *SPL (dB) on the surface of the box calculated using BEM .*

Figure 7 shows the calculated and simulated pressure as a function of frequency at point 1 (see figure 5) near the corner of the box in case of rigid walls in z-direction, soft walls in y-direction with admittance equal to 0.38, and both rigid and soft walls in x-direction.

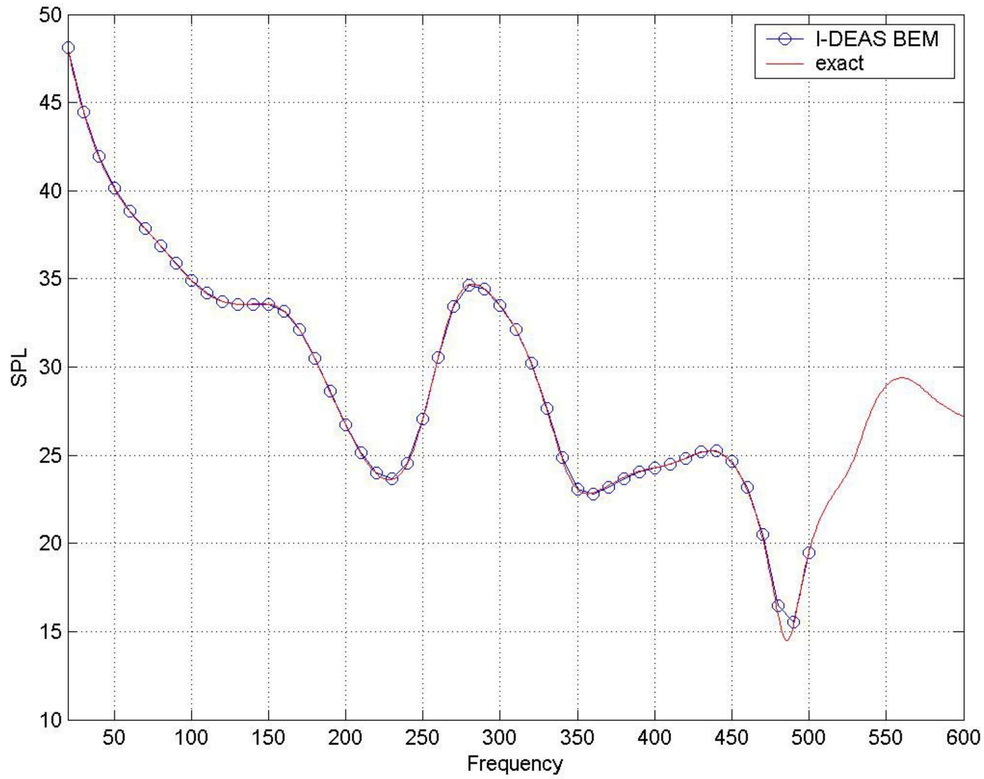

Figure7*: SPL (dB) at point 1 (see figure 5) as a function of frequency. The blue circles correspond to BEM simulation and the red curve corresponds to analytical calculations in a case of a box of three rigid walls and three absorbing walls.*

5. CONCLUSIONS

Traditionally the difficulty to solve eigenvalue equation has hindered the use of general impedance boundary condition in a rectangular box. We have developed a version of Newton iteration method that enables us, in addition to identify the root of caracteristic equation, to specify the ordinal number of the acquired root.
We calculated the frequency response of a point source in a box using two numerical solution algorithms in order to be able to compare and verify the tools and to study the effects of boundary conditions.

6. REFERENCES

[1] Polack, J.-D., *Appl.Acoustics 38 (1993) 235-244*
[2] Morse, P.M., and Ingard, K.U., *Theoretical Acoustics,* Princeton University Press, USA