Direct Simulation for Wind Instrument Synthesis

DAFX 08

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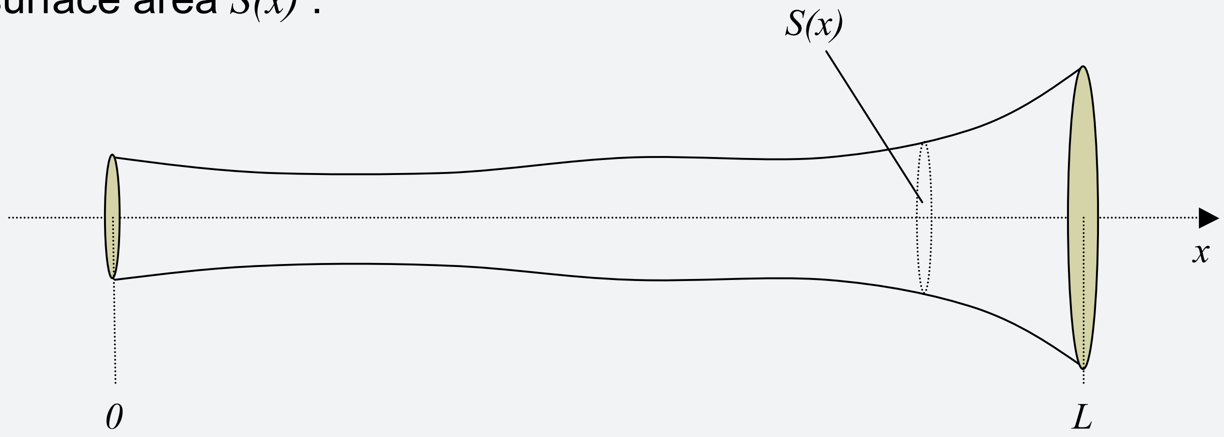
- **Webster's equation**
- **Finite difference schemes**
- **Efficiency, accuracy and stability**
- **Sound examples: Single reed wind instruments**

Webster's Equation

- Usual starting point for wind instrument models (and speech): an acoustic tube, surface area $S(x)$:



- Under various assumptions, velocity potential $\Psi(x,t)$ satisfies:

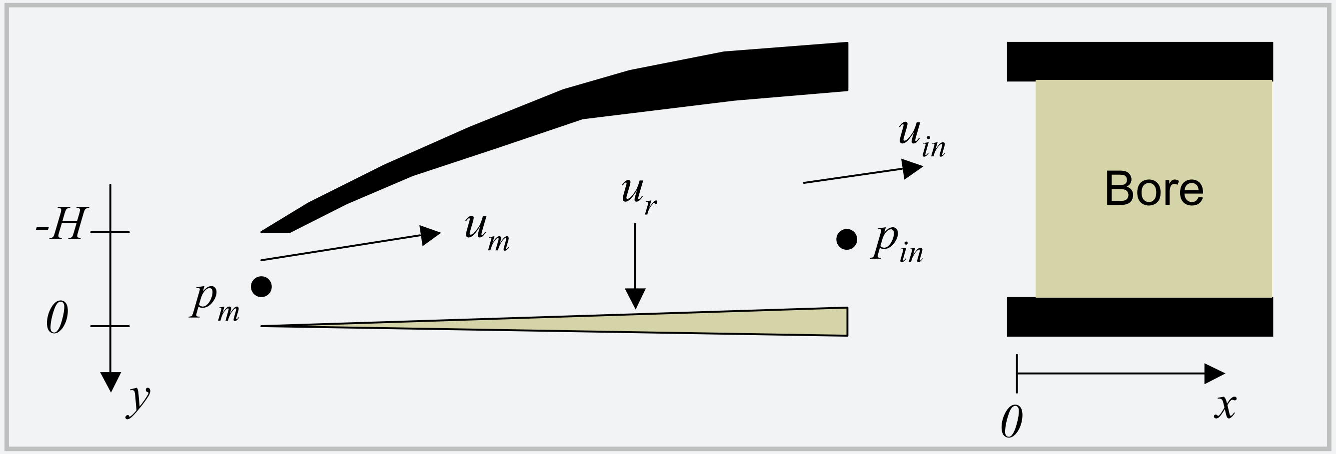
$$\Psi_{tt} = c^2\left(S\Psi_x\right)_x$$

- $\Psi(x,t)$ related to pressure $p(x,t)$ and volume velocity $u(x,t)$ by:

$$p = \rho\Psi_t \qquad u = -S\Psi_x$$

Single Reed Model

- A standard one-mass reed model:



- A driven oscillator:

$$\boxed{\ddot{y} + g\dot{y} + \omega_0^2 y} + \boxed{\omega_1^{1+\alpha}\left([-y-H]^+\right)^\alpha} = \boxed{-a_1 p_\Delta}$$

Linear oscillator terms — Collision term — Driving term

Mouthpiece pressure drop $p_\Delta = p_m - p_{in}$	Flow conservation $u_{in} = u_m - u_r$	Bore coupling $p_{in} = \rho \Psi_t(0,t)$
Flow nonlinearity $u_m = a_2 [y+H]^+ \sqrt{\lvert p_\Delta \rvert}\,\mathrm{sgn}(p_\Delta)$	Flow induced by reed $u_r = a_3 \dot{y}$	$u_{in} = -S(0)\Psi_x(0,t)$

Radiation Boundary Condition

- At the radiating end ($x=L$), an approximate boundary condition is often given in impedance form:

$$P(s) = Z(s)U(s) \qquad\qquad Z(s) = As - Bs^2$$

- Models inertial mass and loss.
- BUT: not positive real → not passive.
- A better approximation (p.r., passive):

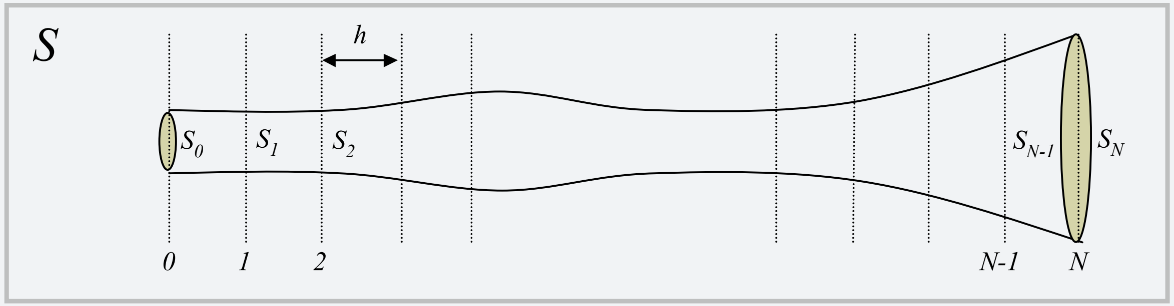
$$Z(s) = \frac{As}{1 + Bs/A}$$

- When converted to the time domain:

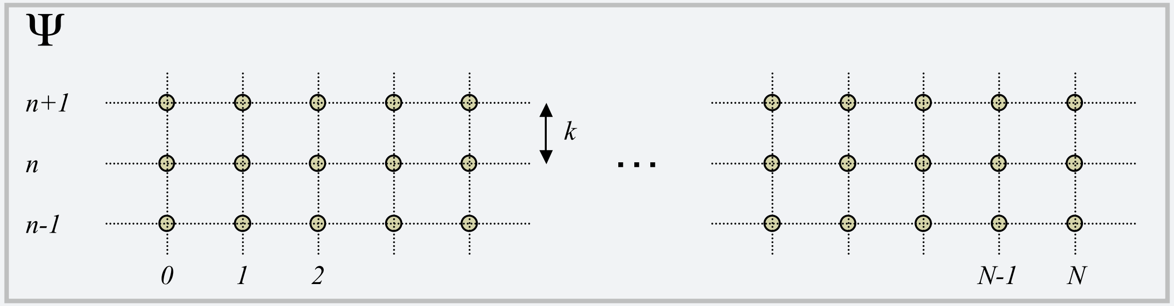
$$\Psi_x + q_1\Psi_t + q_2\Psi = 0 \qquad \text{at} \qquad x = L$$

Finite Difference Scheme

- Sample bore profile at locations
  $x = lh,\ l = 0,\dots,N$
- h = grid spacing



- Introduce grid function Ψ, at locations
  $x = lh,\ l = 0,\dots,N$
  $t = nk,\ n = 0,\dots$
- k = time step



- Here is one particular finite difference scheme (explicit, 2nd order accurate)

$$\Psi_l^{n+1} = 2\lambda^2 \frac{S_l + S_{l+1}}{S_{l+1} + 2S_l + S_{l-1}} \Psi_{l+1}^n + 2\lambda^2 \frac{S_l + S_{l-1}}{S_{l+1} + 2S_l + S_{l-1}} \Psi_{l-1}^n + 2\left(1 - \lambda^2\right)\Psi_l^n - \Psi_l^{n-1}$$

- Courant number λ defined as $\lambda = ck/h$

Stability and Special Forms

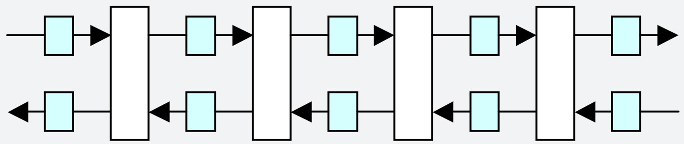
- Can show (energy methods) that scheme is stable, over interior, when

$$\lambda \leq 1$$

- When λ = *1*, scheme simplifies to:

$$\Psi_l^{n+1} = 2\frac{S_l + S_{l+1}}{S_{l+1} + 2S_l + S_{l-1}}\Psi_{l+1}^n + 2\frac{S_l + S_{l-1}}{S_{l+1} + 2S_l + S_{l-1}}\Psi_{l-1}^n - \Psi_l^{n-1}$$

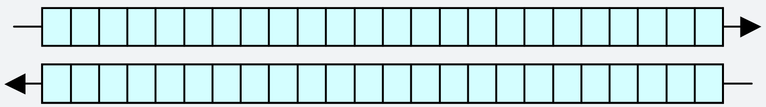
…equivalent to Kelly-Lochbaum scattering method



- When λ = *1*, and S = const., scheme simplifies further:

$$\Psi_l^{n+1} = \Psi_{l+1}^n + \Psi_{l-1}^n - \Psi_l^{n-1}$$

…equivalent to digital waveguide (exact integrator)



Stability Condition and Tuning

- Stability condition requires $\lambda \leq 1 \quad \rightarrow \quad h \geq ck$
- For simplicity, would like to choose an h which divides L evenly, i.e.,

$$L/h = N \qquad \text{for integer} \quad N$$

- Not possible for waveguide/Kelly-Lochbaum methods --- $h=ck$. Result: detuning, remedied using fractional delays.
- In an FD scheme, can choose h as one wishes. Result: very minor dispersion/loss of audio bandwidth. Numerical cutoff:

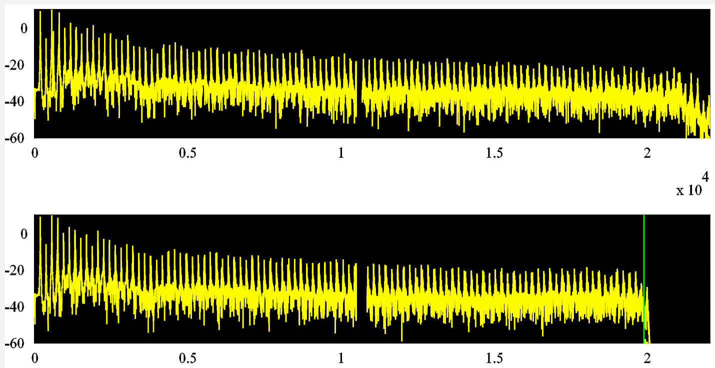
$$f_c = \frac{f_s}{\pi}\sin^{-1}(\lambda) \leq \frac{f_s}{2}$$

- Worst case near $f_s = 44.1$ kHz, typical wind instrument dimensions:

$$f_c \cong 20 \quad \text{kHz}$$

$f_s \cong 44037 \quad \text{Hz}$

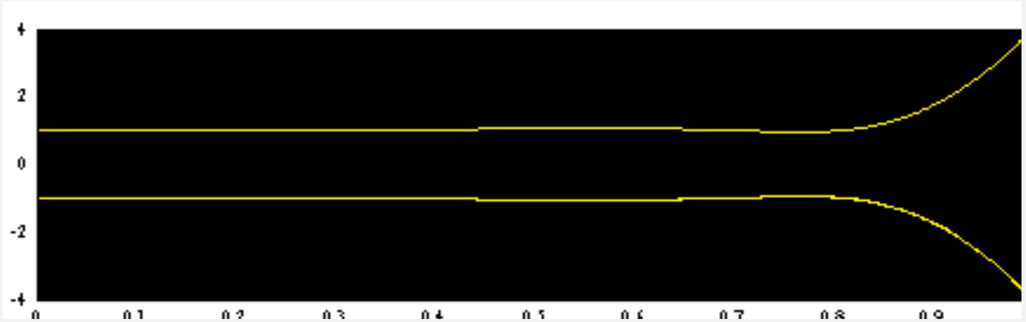
$f_s \cong 44036 \quad \text{Hz}$



Accuracy—Modal Frequencies

- Numerical dispersion---normally a problem for FD schemes!
- This is a 2nd order scheme---might expect severe mode detunings…
- Not so…

E.g., for a lossless clarinet bore…

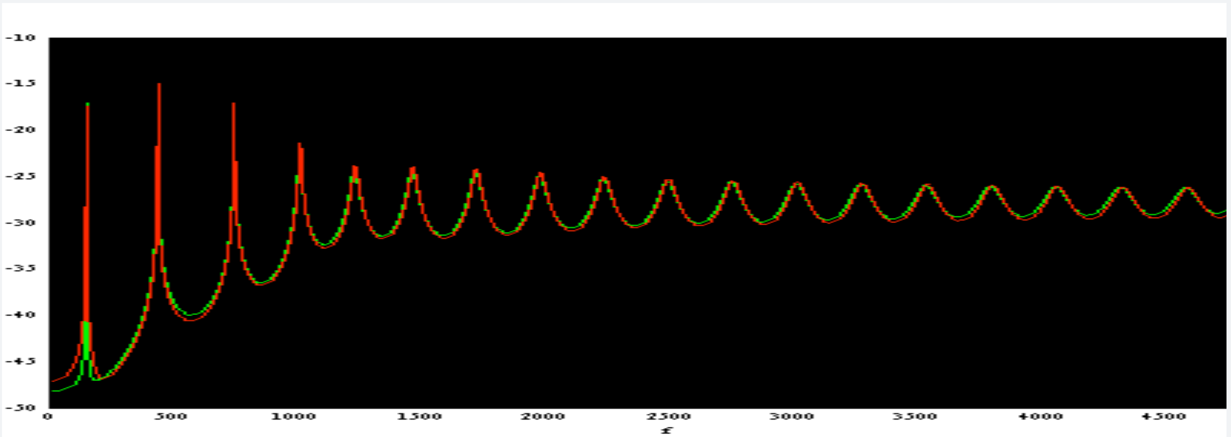


…calculated modal frequencies are nearly exact, over the entire spectrum

Mode #	Freq. (FD, Hz)	Freq. (exact, Hz)	cent diff.
1	141.89	141.96	0.86
2	413.79	413.95	0.65
3	705.55	705.55	0.00
12	3144.04	3142.63	- 0.77

Accuracy—Transfer Impedance

- Even under more realistic conditions (i.e., with radiation loss), behaviour is extremely good:
- Transfer impedance (mouth → radiating end):



Red: exact (calculated at 400 kHz) **Green: calculated at 44.1 kHz**

- Upshot: FD approximation converges very rapidly…
- …"perceptually" exact, even at audio sample rate.
- No compelling reason to look for better schemes…

Explicit Updating

- Discretization of oscillator: $$\ddot{y} + g\dot{y} + \omega_0^2 y + \omega_1^{1+\alpha}\left([-y-H]^+\right)^{\alpha} = -a_1 p_\Delta$$

Parameterized implicit discretization

Implicit discretization

$$y^{n+1} - 2y^n + y^{n-1} + \frac{gk}{2}\left(y^{n+1} - y^{n-1}\right) + \boxed{\omega_0^2 k^2 \theta y^n + \frac{\omega_0^2 k^2 (1-\theta)}{2}\left(y^{n+1} + y^{n-1}\right)} - \boxed{\frac{\omega_1^{\alpha+1} k^2}{2}\left(y^{n+1} + y^{n-1}\right)\left([-y^n - H]^+\right)^{\alpha-1}} = -a_1 p_\Delta^n$$

Exact integrator possible for linear part of oscillator…

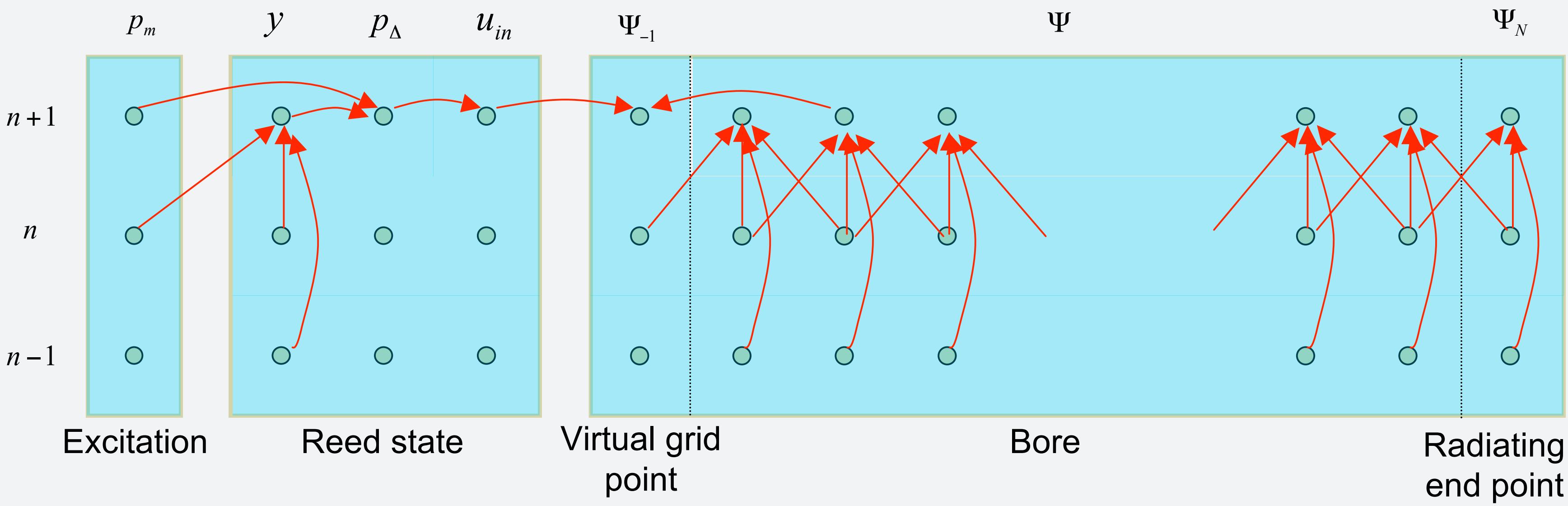
Explicit update…

Mouthpiece pressure drop	Flow conservation	Bore coupling
$p_\Delta^n = p_m^n - p_{in}^n$	$u_{in}^n = u_m^n - u_r^n$	$p_{in}^n = \frac{\rho}{2k}\left(\Psi_0^{n+1} - \Psi_0^{n-1}\right)$
Flow nonlinearity	Flow induced by reed	
$u_m^n = a_2[y^n + H]^+ \sqrt{\lvert p_\Delta^n \rvert}\,\mathrm{sgn}(p_\Delta^n)$	$u_r^n = \frac{a_3}{2k}\left(y^{n+1} - y^{n-1}\right)$	$u_{in}^n = -\frac{S_0}{2h}\left(\Psi_1^n - \Psi_{-1}^n\right)$

- Implicit discretization → excellent stability properties
- Unknowns always appear linearly…

Explicit Updating

- Can find a flow path in order to update all the state variables (sequentially)



- Similar to setting of “reflection-free port resistances” in linear WDF networks…
- …but more general.

Note on Stability

- The scheme for the bore + bell termination, in isolation, is guaranteed stable.
- Situation more complicated when reed mechanism is connected.
- Consider system under transient conditions (input $p_m = 0$):

$$\underbrace{H(t)}_{\text{Total energy}} = \underbrace{H_{bore}(t)}_{\text{Stored energy in bore}} + \underbrace{H_{bell}(t)}_{\text{Stored energy at bell}} + \underbrace{H_{reed}(t)}_{\text{Stored energy of reed}} \le \underbrace{H(0)}_{\text{Initial stored energy}}$$

True…

- System is dissipative → state bounded for any initial conditions.
- Under forced conditions, would like:

$$\boxed{\xcancel{\underbrace{H(t)}_{\text{Total energy}} \le \underbrace{H(0)}_{\text{Initial stored energy}} + \underbrace{\int_0^t f(p_m)\,dt'}_{\text{Energy supplied externally}}}}$$

Unfortunately this is false…

- Upshot: impossible to bound solutions of model system under forced conditions
- Best one can do: ensure energy balance is respected in FD scheme…

Computational Cost

For a given sample rate f_s, bore length L, and wave speed c, the computational requirements are:

- $2Lf_s/c$ units memory
- $4Lf_s^{\,2}/c \rightarrow 6Lf_s^{\,2}/c$ flops/sec.

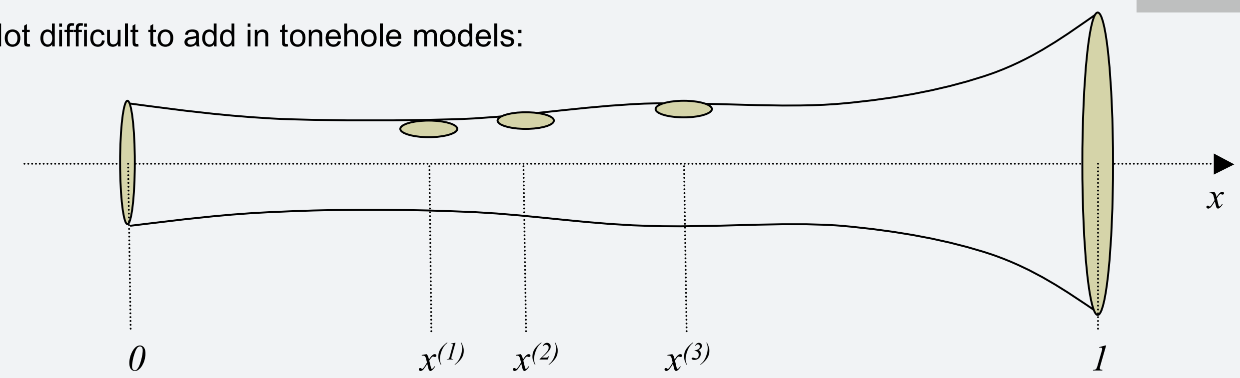
…independent of bore profile. Reed/tonehole/bell calculations are $O(1)$ extra ops/memory per time step

Example: clarinet → 15 Mflops/sec., at $f_s = 44.1$ kHz

Not a lot by today's standards…far faster than real time.

Toneholes

- Not difficult to add in tonehole models:



- Can add terms pointwise to Webster's equation:

State of tonehole q

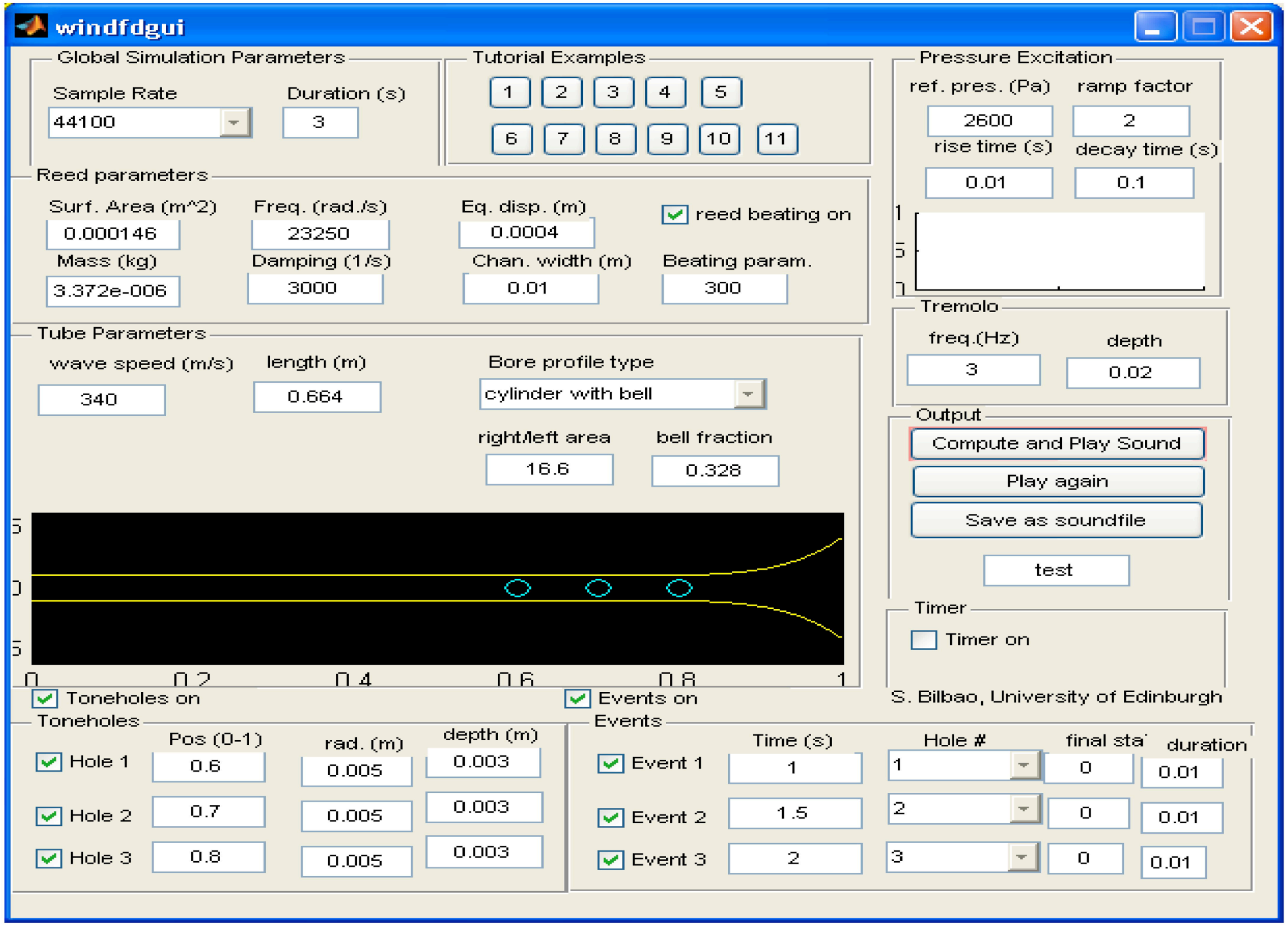
$$\Psi_{tt} = c^2 \left(S\Psi_x\right)_x + \sum_{q=1}^{M} \delta\left(x - x^{(q)}\right) m^{(q)}$$

$$m^{(q)} = A\Psi_{tt}(x^{(q)}, t) + B\Psi(x^{(q)}, t) \qquad q = 1, \ldots, M$$

Physical parameters defining tonehole q

- In FD implementation, can be added anywhere along bore (Lagrange interpolation):

GUI: Matlab


windfdgui
Global Simulation Parameters
Sample Rate
44100
Duration (s)
3
Tutorial Examples
1 2 3 4 5
6 7 8 9 10 11
Pressure Excitation
ref. pres. (Pa)
2600
ramp factor
2
rise time (s)
0.01
decay time (s)
0.1
Reed parameters
Surf. Area (m^2)
0.000146
Freq. (rad./s)
23250
Eq. disp. (m)
0.0004
reed beating on
Mass (kg)
3.372e-006
Damping (1/s)
3000
Chan. width (m)
0.01
Beating param.
300
Tremolo
freq.(Hz)
3
depth
0.02
Tube Parameters
wave speed (m/s)
340
length (m)
0.664
Bore profile type
cylinder with bell
right/left area
16.6
bell fraction
0.328
Output
Compute and Play Sound
Play again
Save as soundfile
test
Timer
Timer on
0 0.2 0.4 0.6 0.8 1
Toneholes on
Events on
S. Bilbao, University of Edinburgh
Toneholes
Pos (0-1)
rad. (m)
depth (m)
Hole 1 0.6 0.005 0.003
Hole 2 0.7 0.005 0.003
Hole 3 0.8 0.005 0.003
Events
Time (s)
Hole #
final sta
duration
Event 1 1 1 0 0.01
Event 2 1.5 2 0 0.01
Event 3 2 3 0 0.01

Sound Examples

- Clarinet:
- Saxophone:
- Multiphonics/squeaks:

Conclusion

- Disadvantages:
  - Costs more to compute than a typical waveguide model (but still not much!)
- Advantages:
  - Bore modeling becomes trivial…
  - More general extensions possible (NL wave propagation)
  - Far more design freedom that, e.g., WG/WD methods