AN AUDIO MOTIVATED HYBRID OF WARPING AND KAUTZ FILTER TECHNIQUES

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Key concepts

Kautz filters – a special class of pole-zero (IIR) filters, forced structurally to produce orthonormal tap output impulse responses - rational orthonormal basis representations for signals and systems - a generalization of the z-transform

Frequency warping – a dispersive signal transformation, corresponding to an one-to-one conformal mapping of the z-transform representation - a method for producing frequency responses on a warped non-uniform frequency scale

What's really so novel?

- The utilization of Kautz filters in challenging pure filter synthesis by modern DSP means in design and implementation
- The proposed method for the optimization of Kautz filter poles - or more generally, a new IIR filter design tool
- The use of an intermediate warping procedure in the pole optimization to allocate desired frequency resolution

What is achieved?

- Efficient modelling of complicated audio responses - sharp focusing on distinct resonances with accurate phase as well as magnitude modelling
- Benefits of the orthonormality - explicit control of the modelling error, trivial model reduction - simultaneous time- and frequency-domain design
- An additional design parameter trough the warping step - e.g, detailed models for the low-frequency region

Kautz functions and filters

Kautz filters originate from rational orthonormal functions [10]

$$G_i(z) = \frac{\sqrt{1 - z_i z_i^*}}{1 - z_i z^{-1}} \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots, \qquad (1)$$

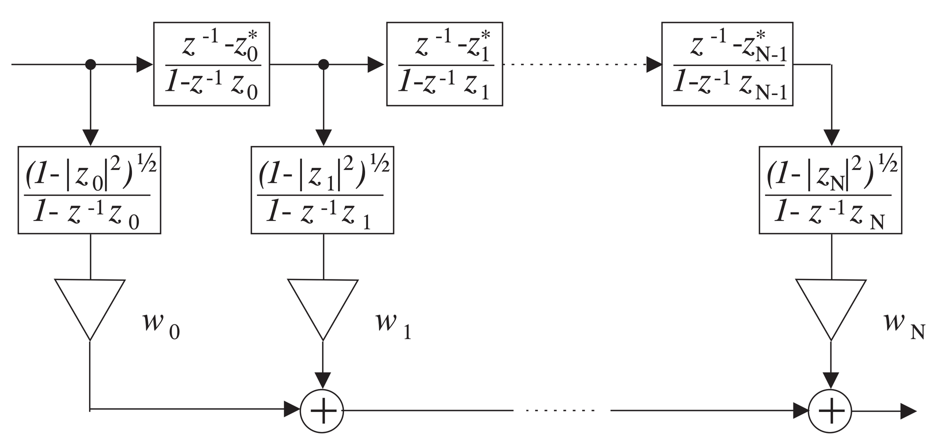
defined by some set of points $\{z_i\}_{i=0}^{\infty}$ in the unit disk. Functions (1) have a recurrent structure, i.e., a (finite) weighted sum of functions (1) form a transversal filter

$$K(z, N, \mathbf{z}, \mathbf{w}) = \sum_{i=0}^{N} w_i G_i(z) = \sum_{i=0}^{N} w_i \frac{\sqrt{1 - z_i z_i^*}}{1 - z_i z^{-1}} A_i(z, \mathbf{z}),$$

defined by poles $\mathbf{z} = [z_0 \ \cdots \ z_N]^T$ and tap-output weights $\mathbf{w} = [w_0 \ \cdots \ w_N]^T$, where the transversal part is a tapped all-pass chain

$$A_i(z, \mathbf{z}) = \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots, N.$$

In agreement with the continuous-time counterpart [6], $K(z, N, \mathbf{z}, \mathbf{w})$ is called a Kautz filter, depicted as



Properties and interpretations of the Kautz filter

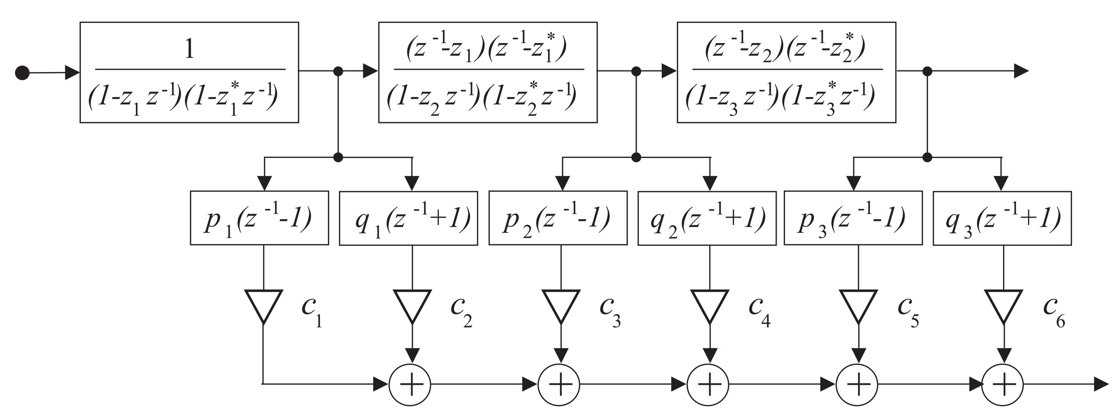
- Causal and stable for $|z_i| < 1$ and any choice of $\{w_i\}$
- Orthonormality - for the tap-output impulse responses, $(g_i, g_k) := \sum_{n=0}^{\infty} g_i(n) g_k^*(n) = 0$ for $i \neq k$, and $(g_i, g_i) = 1$
- Special cases of the Kautz filter:
  - For $z_i = 0$ it degenerates to an FIR filter
  - For $z_i = a, -1 < a < 1$, it is a Laguerre filter [7]
  - Generalized orthonormal basis functions by Heuberger[5]
- When equipped with (Kautz-Fourier) weights $c_i = (h, g_i)$:
  - An orthonormal (Fourier) series expansion - a generalization of the z-transform - a complete basis representation, defining Fourier transforms for any finite-energy h or H:

    $$h = \sum c_i g_i \ \leftrightarrow \ (h, g_i) = c_i = (H, G_i) \ \leftrightarrow \ \sum c_i G_i = H$$
  - Orthogonal projections - truncated series expansions,

    $$\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n) \quad or \quad \hat{H}(z) = \sum_{i=0}^{N} c_i G_i(z), \qquad (2)$$

    taps are independent of ordering and approximation order

- The real-valued Kautz filter for complex conjugate poles [2] - prevents dealing with complex signals and weights:



- $p_i = \sqrt{(1-\rho_i)(1+\rho_i-\gamma_i)/2}$ and $q_i = \sqrt{(1-\rho_i)(1+\rho_i+\gamma_i)/2}$, where $\gamma_i = -2RE\{z_i\}$ and $\rho_i = |z_i|^2$
- A mixture of structures is used for both real and cc-poles

Pole position optimization – the BU-method

Our choice of Kautz filter parametrization is the orthonormal expansion coefficients, as in (2), since the contribution of each chosen pole to the approximation error

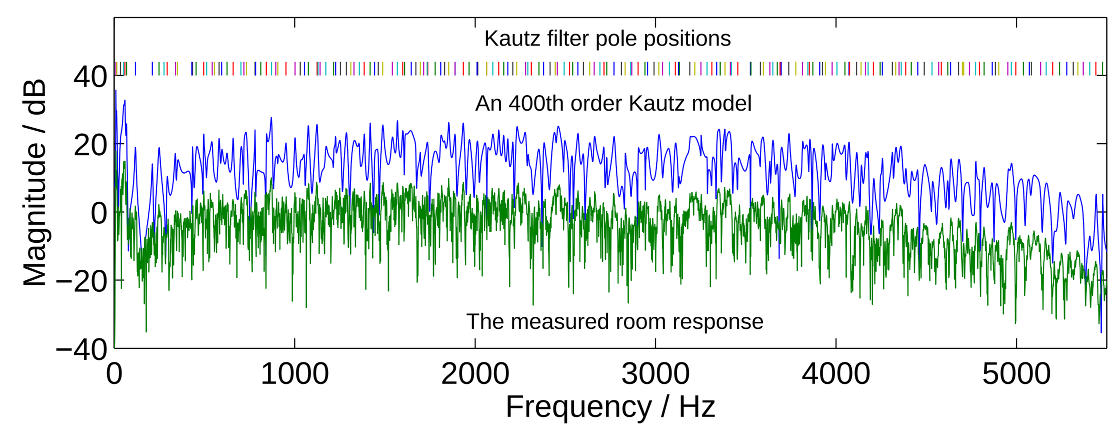
$$\mathcal{E} = \mathcal{H} - \sum_{i=0}^{N} |c_i|^2, \qquad \mathcal{H} = (h, h), \qquad (3)$$

is explicitly at hand. The "only" design problem is then how to choose the poles. Very few attempts has been made to solve this complicated task:

- Methods that restrict to structures with identical sub-blocks, e.g., the Laguerre filter and the two-pole Kautz filter [3, 9]
- A direct adaptive gradient search [4] and an iterative method based on a linearization of the optimization problem [8]

The latter two methods are based on an old concept of complementary signals [11], which states that minimization of (3) is equivalent to an optimization criterion related solely on the all-pass operator $A_N(z)$ defining the Kautz filter. Without reference to [11] or to Kautz filters, Brandenstein and Unbehauen deduce the same optimization criterion for the determination of the denominator in pure FIR-to-IIR filter conversion - our modification and adoption to the context of Kautz filters is named the BU-method.

The BU-method is capable of producing very large sets of accurate poles for challenging target responses:



The warped BU-method

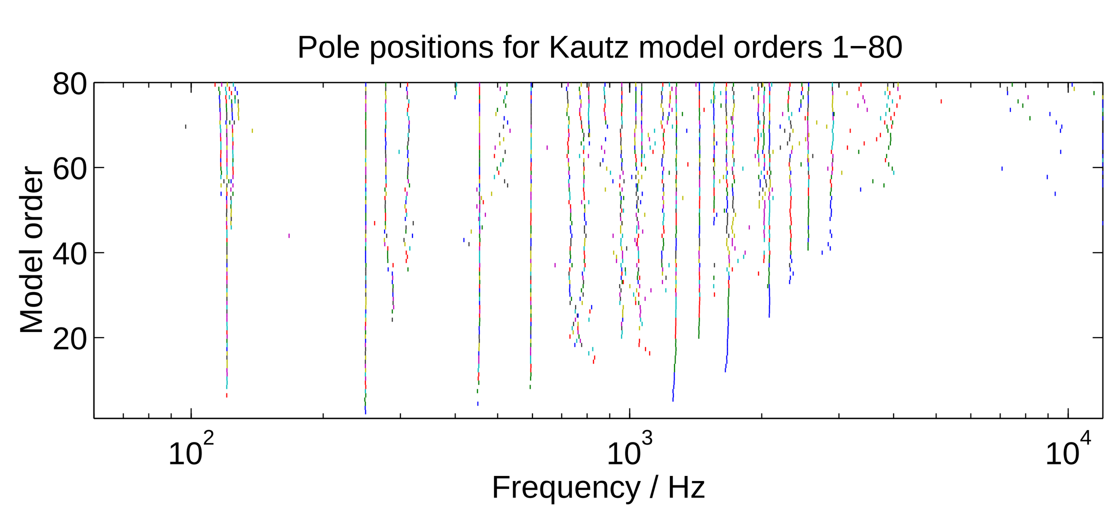
The BU-method operates directly on time-domain responses - desired characteristics can be emphasized in the pole generation using target response manipulations. Here we demonstrate the effect of intermediate frequency warping. Laguerre-warping is used since it is the true orthonormal transformation:

- Chose a warping parameter a, $-1 < a < 1$
- Warp the target response h(n) - a simple implementation: feed $h(-n)$ to a Kautz filter with $z_i = a$ and read the tap-outputs $x_i(n) = G_i[h(-n)]$ at $n = 0$: $\tilde{h}(i) = x_i(0)$
- Generate BU-poles using $\tilde{h}$ and some model order N
- Map the pole set according to $z \mapsto (z + a)/(1 + az)$
- Evaluate $c_i = (h, g_i)$ and compose $\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n)$

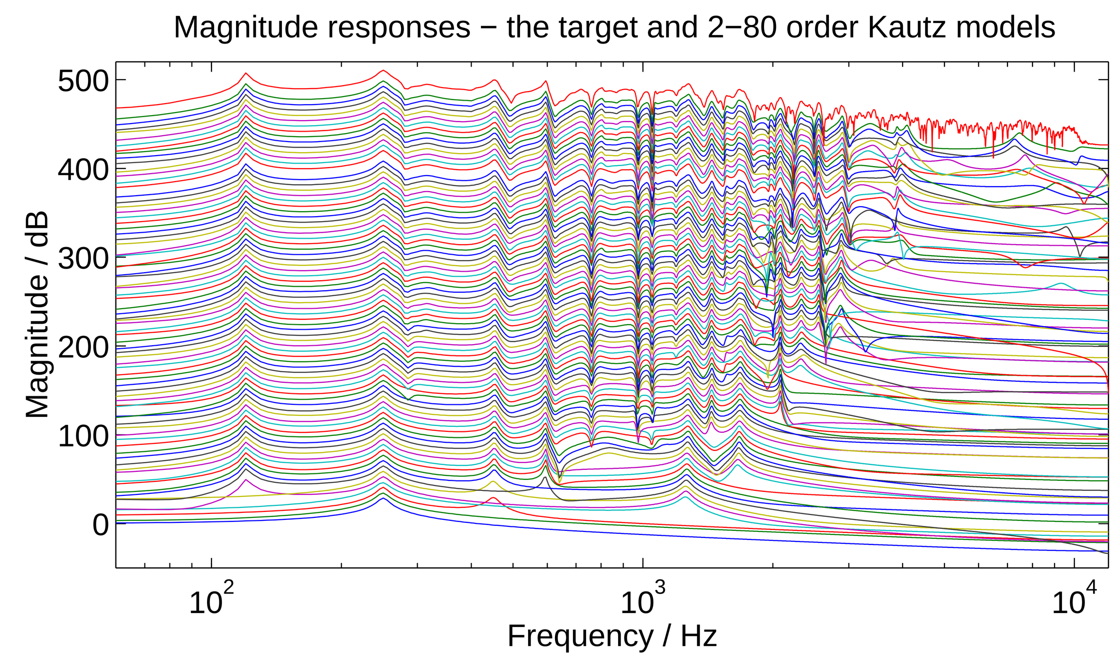
Some illustrative examples

In the following examples a measured acoustic guitar body impulse response is modelled using various warpings and Kautz model orders.

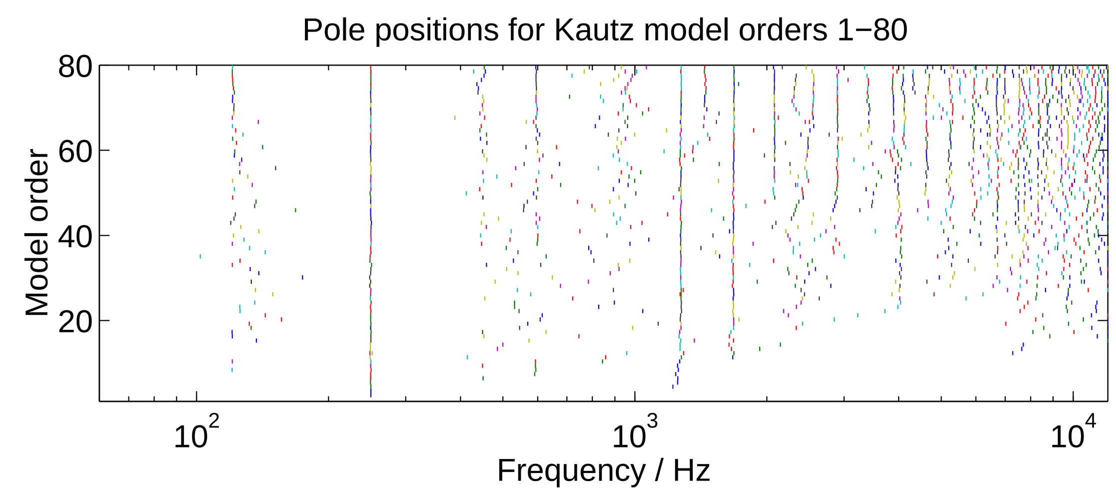
- Warped BU-poles ($a = 0.7$) with respect to the model order:

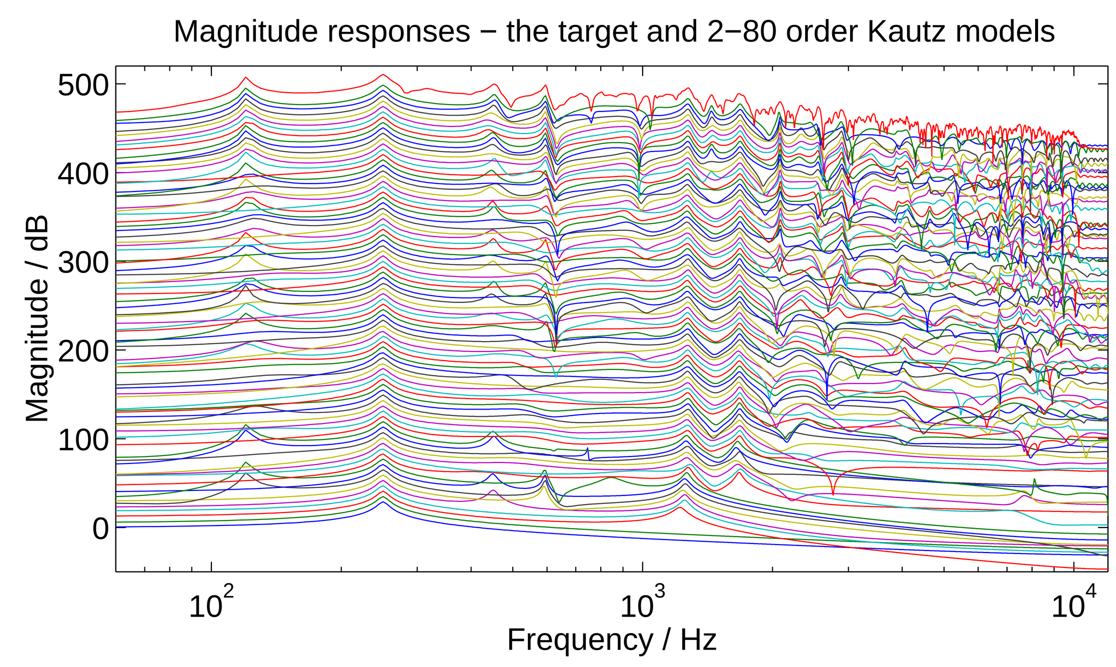


- Corresponding Kautz model magnitude responses, along with the target magnitude response at the top:

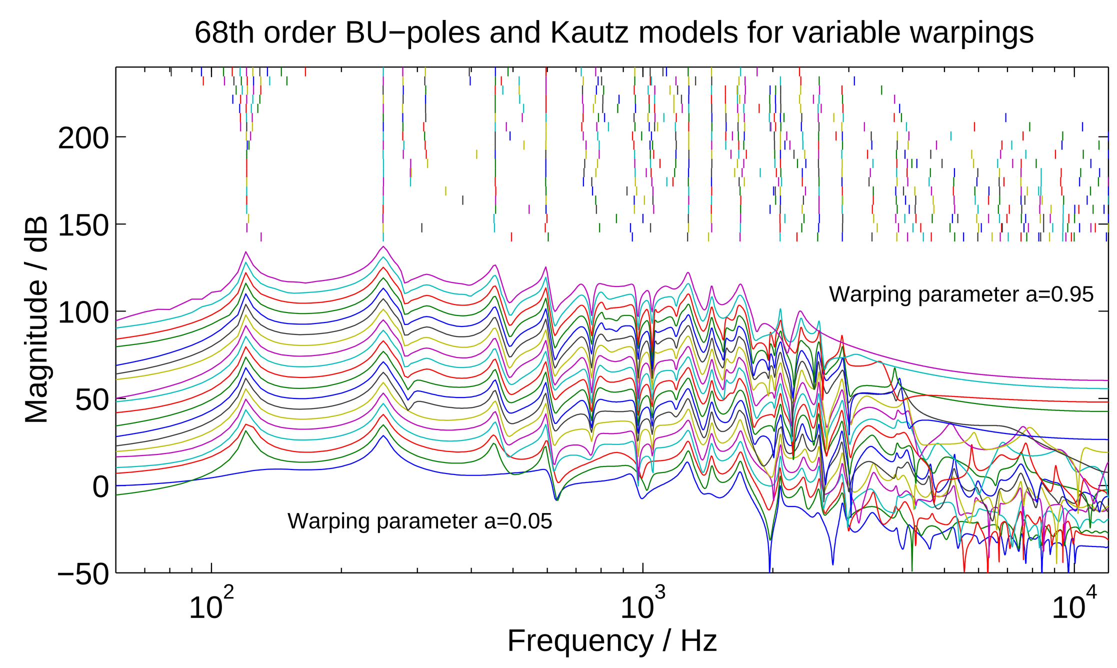


- For comparison, the same setup for the un-warped case:





- A fixed model order 68 is chosen and the effect of warping is demonstrated using warping parameters $a = 0.05 - 0.95$ with steps of 0.05:



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