Modeling and Equalization of Audio Systems Using Kautz Filters

Tuomas Paatero, Matti Karjalainen, and Aki Härmä

Helsinki University of Technology
Laboratory of Acoustics and Audio Signal Processing
P. O. Box 3000, FIN-02015 HUT, Finland
tuomas.paatero@hut.fi

Kautz filters in a nutshell

Kautz filters [6] are a class of fixed-pole linear-in-parameter IIR filters

$$K(z, N, \mathbf{z}, \mathbf{w}) = \sum_{i=0}^{N} w_i G_i(z) = \sum_{i=0}^{N} w_i \frac{\sqrt{1 - z_i z_i^*}}{1 - z_i z^{-1}} A_i(z, \mathbf{z}),$$

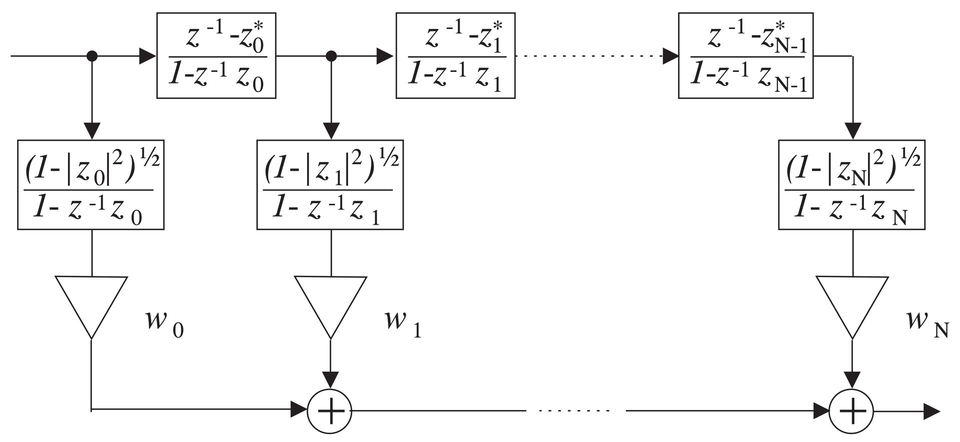
$\mathbf{z} = [z_0 \; \cdots \; z_N]^T$, $\mathbf{w} = [w_0 \; \cdots \; w_N]^T$, composed of a transversal all-pass backbone

$$A_i(z, \mathbf{z}) = \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots, N,$$

and all-pole tap-output filters, forced to produce orthonormal tap-output impulse responses, and originating from rational orthonormal (basis) functions [10]

$$G_i(z) = \frac{\sqrt{1 - z_i z_i^*}}{1 - z_i z^{-1}} \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots, \tag{1}$$

defined by any set of points $\{z_i\}_{i=0}^{\infty}$ in the unit disk. A particular Kautz filter is thus determined by a set of (stable) poles $\{z_i\}_{i=0}^{N}$ and somehow assigned filter coefficients $\{w_i\}_{i=0}^{N}$:

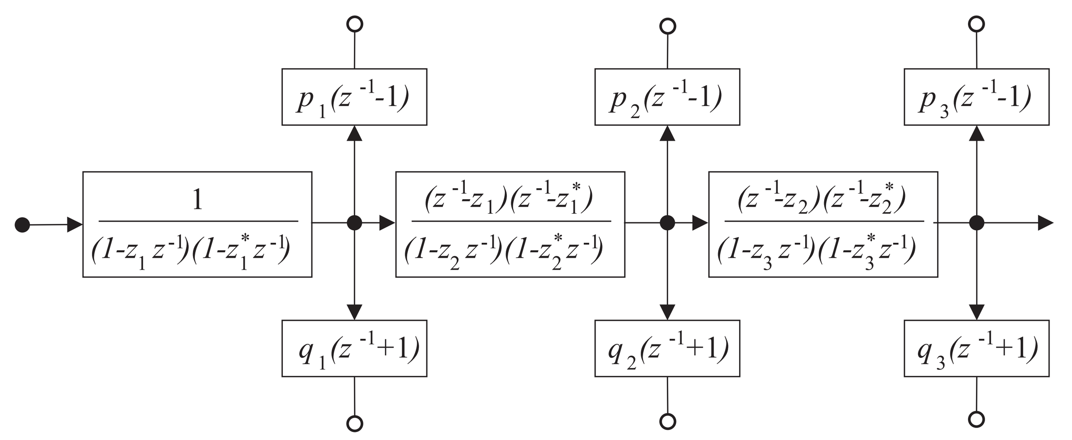


Some more or less familiar special cases:

- for $z_i = 0$ it degenerates to an FIR filter
- for $z_i = a, -1 < a < 1$, it is a Laguerre filter [7] where the tap-filters can be replaced by a common pre-filter
- Generalized Orthonormal Basis Functions [5] associated with a recurrent sequence of poles

The real-valued Kautz filter

A Kautz filter produces real tap-output signals only in the case of real poles. However, from a sequence of real or complex conjugate poles it is always possible to form modified real Kautz structures. For example, for purely complex (conjugate) poles we may choose [2]



This solution is not unique, but it is simple and intuitive:

- second-order section outputs are orthogonal
- from which an orthogonal tap-output pair is formed
- and normalized using:
  - $p_i = \sqrt{(1-\rho_i)(1+\rho_i-\gamma_i)/2}$,
  - $q_i = \sqrt{(1-\rho_i)(1+\rho_i+\gamma_i)/2}$,

  where $\gamma_i = -2RE\{z_i\}$ and $\rho_i = |z_i|^2$ can be recognized as corresponding second-order polynomial coefficients

We use an obvious mixture of first- and second-order sections for sets of both real and complex conjugate poles.

The Kautz model for signals and systems

Kautz filters provide linear-in-parameter models for many types of system identification and approximation schemes. Here too, we have various interpretations, criteria, and methods for the model parametrization. The "prototype" least-square approaches are implied by the signal space descriptions:

Approximation: A basis representation of any causal and finite-energy signal is obtained as its Fourier series expansion with respect to functions (1)

Input-output-data identification: Tap-output signals span an "approximation space" for any causal and stable system – normal equations assembled from correlation terms provide least-square optimal model parametrizations

Furthermore, here we restrict to the approximation of a given target response $h(n)$ with truncated Fourier series expansions

$$\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n), \quad c_i = (h, g_i),$$

where functions $\{g_i(n)\}$ are impulse responses or inverse z-transforms of functions (1). We choose these true orthonormal expansion coefficients, because:

- Fourier coefficients are easily obtained by feeding $h(-n)$ to the Kautz filter and reading the tap-outputs $x_i(n) = G_i[h(-n)]$ at $n = 0$: $c_i = x_i(0)$,
- which can be seen as a generalization of rectangular window FIR design,
- rendering a direct view to error evaluation and model reduction,
- coefficients being independent of ordering and approximation order,
- providing implicitly simultaneous time and frequency domain design,
- and powerful means to the Kautz filter pole position optimization.

Choosing of the Kautz filter poles

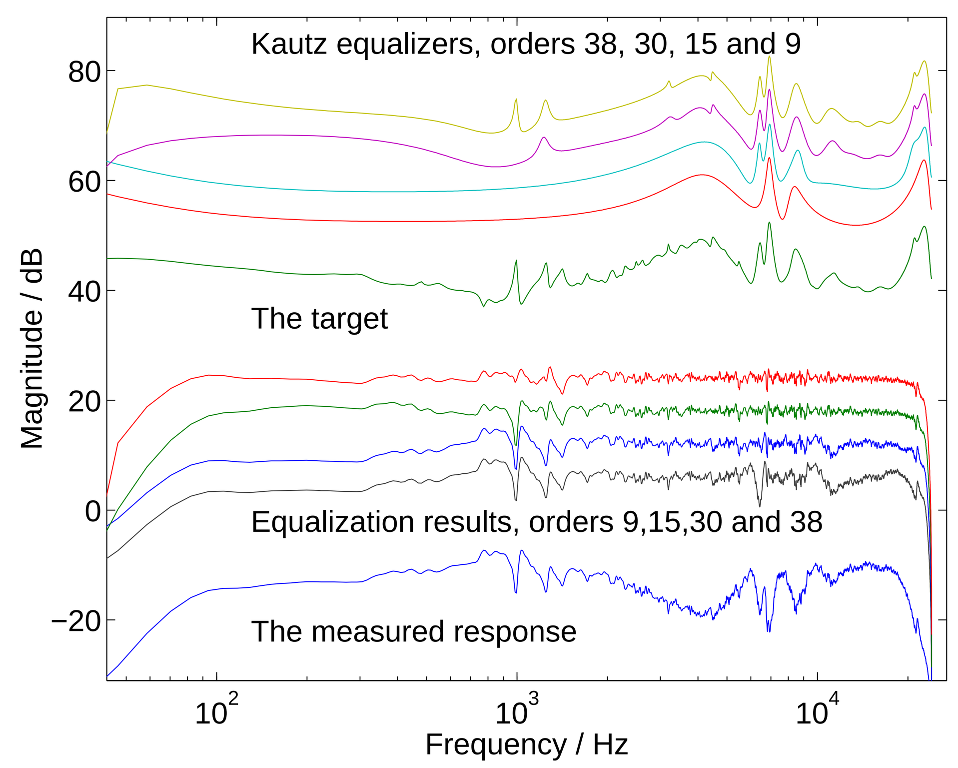
There are various strategies in search for suitable pole sets, for example:

- fixed pole distributions or repetitive use of a small subset of poles
- sophisticated guesses, and random or iterative search
- manual tuning of the poles to the target response resonances
- indirect means, such as all-pole or pole-zero modeling
- a relation between optimal model parameters and error energy surface stationary points [3] or a classification of systems [9]
- direct gradient [4] or iterative [8] optimization of the structure, based on the *complementary division* of signal energy [11]

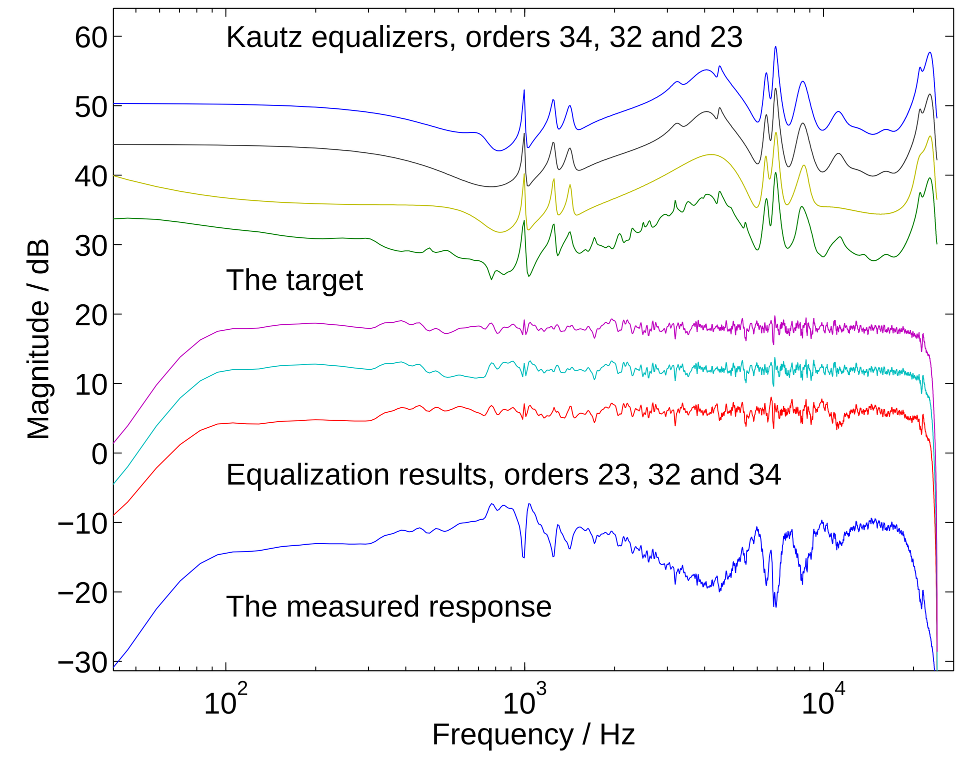
Related to the last-mentioned, we have adopted a method proposed originally to pure FIR-to-IIR filter conversion [1], to the context of Kautz filter pole optimization. It resembles the *Steiglitz-McBride method* of pole-zero modeling, but it genuinely and effectively optimizes the pole positions of a real Kautz filter, producing unconditionally stable and (theoretically globally) optimal pole sets for a desired filter order. In the following we use this *BU-method* as such or combined with, e.g., *warped design* or manual choosing and tuning of poles.

Audio oriented example 1: Loudspeaker equalization

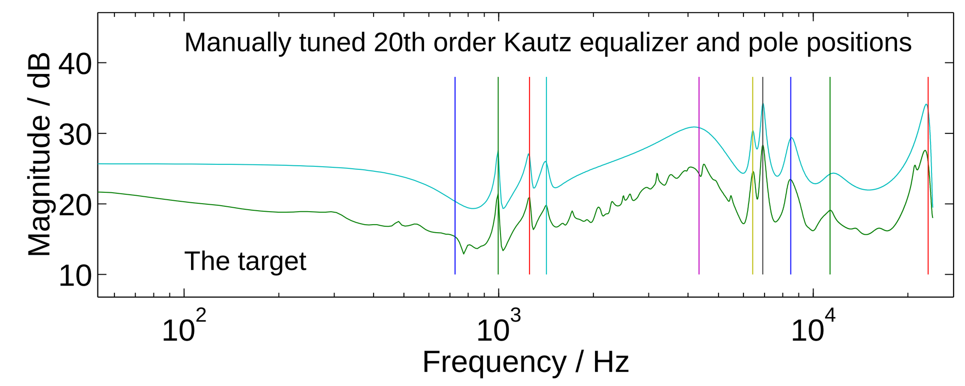
Here we demonstrate the use of Kautz filters in pure magnitude equalization, based on an inverted target response. Direct utilization of the BU-method provides, e.g., following results:



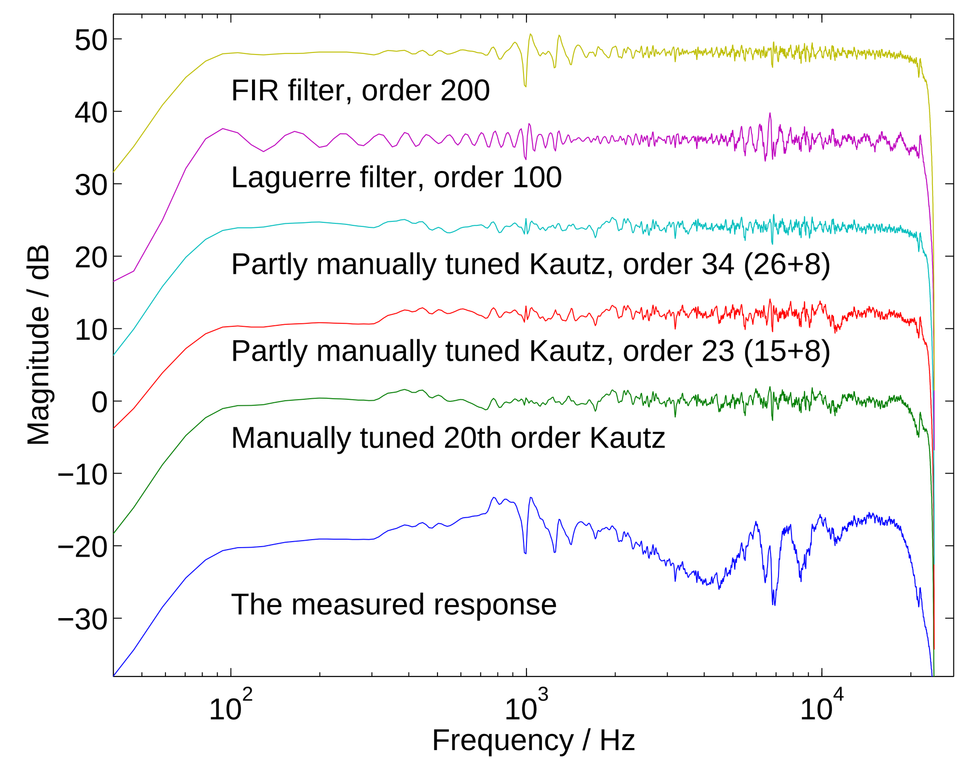
To improve the modeling at 1 kHz, we add 3–4 manually tuned pole pairs to the BU-pole sets, omitting some undesired poles, producing



We may also tune all the poles manually, e.g., with 10 distinct pole pairs, chosen and tuned to fit the magnitude response:

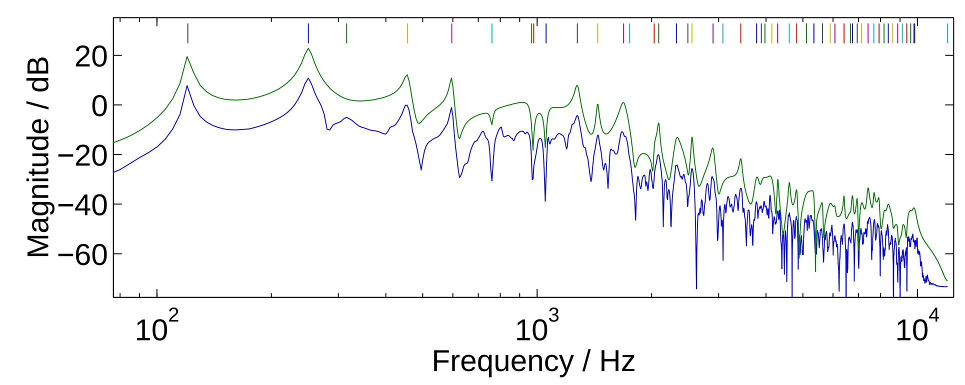


A comparison of FIR, Laguerre, and Kautz equalization results:

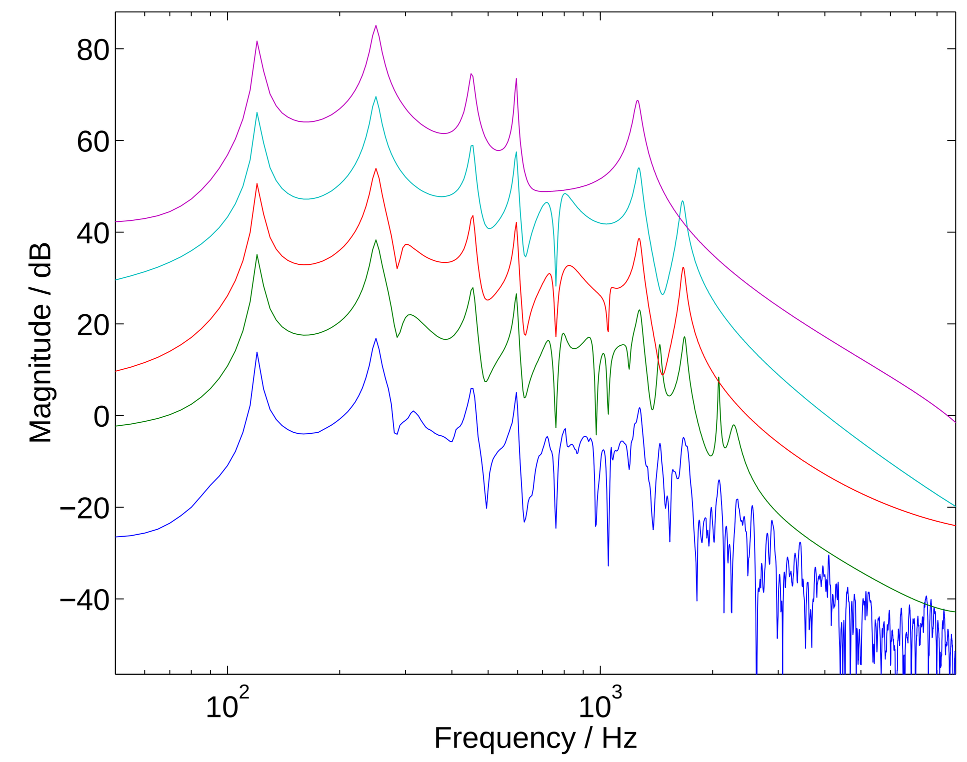


Audio oriented example 2: Modeling of a guitar body response

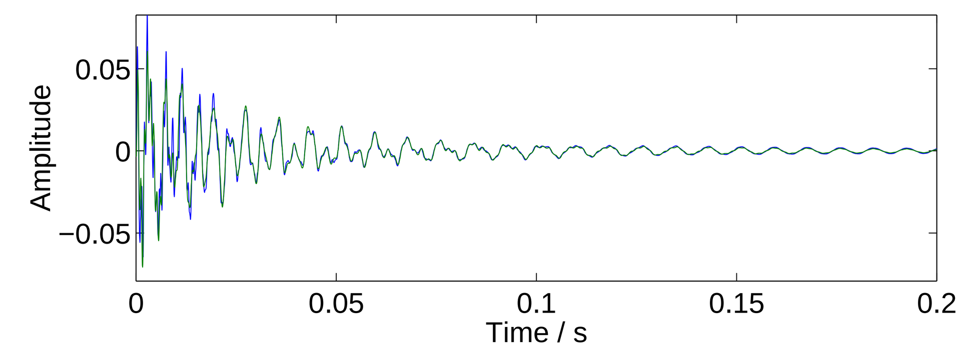
As another example of Kautz modeling we approximate a measured acoustic guitar body response. The BU-method is able to capture essentially the whole resonance structure at filter order 102, obtained by pruning a 120th order BU-pole set:



Lines indicate pole pair positions. Especially in this case of a target response dominated by the low-frequency part, we may compose very low order Kautz models by applying the BU-method to the *warped target response*, and by mapping the produced poles back to the original frequency domain. Displayed with offset from top to bottom, Kautz models of orders 10, 16, 20 and 40, and the target magnitude response:



The warped BU-method finds the five prominent resonances at filter order 10, in contrast to the unwarped design, where the required order is about 100! Finally, we demonstrate that good fit to the five prominent resonances of the 10th order Kautz filter means also good match in the time-domain to the measured response:



Further information and acknowledgements

A more thorough presentation of the underlying theory and the merely stated results can be found in the *Proceedings* or in other related `~/publications` at `http://www.acoustics.hut.fi` as well as MATLAB scripts and demos in `~/software/kautz`. This work has been supported by the Academy of Finland as a part of the project "Sound source modeling".

References

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