Answers to exercise 10

1. (a) In matrix addition elements are added element-wise, so we can restrict our analysis to simple elements. Addition $\mathbf{A}+\mathbf{B}$ is symmetric to $\mathbf{A}$ and $\mathbf{B}$ so we need to study an error in an element of $\mathbf{A}$ only. Let a be an element of $\mathbf{A}$ and b and element of $\mathbf{B}$. Then $a+b=c$. If a has an error Δa, then $a+\Delta a+b=c+\Delta c$.

   The absolute condition is

$$\kappa('+', a) = \sup_{a,\Delta a} \frac{|a+b-(a+\Delta a+b)|}{|\Delta a|} = \sup_{a,\Delta a} \frac{|\Delta a|}{|\Delta a|} = 1.$$

   (b) Matrix multiplication is, for the individual elements merely vector inner-products, so we can restrict our analysis to inner-products only.

   The inner-product is defined as

$$c = \mathbf{a}^H \mathbf{y} = \sum_{k=1}^{n} a_k y_k. \tag{1}$$

   Algorithmically, it can be calculated with:

```
s(1) = 0
for k = 1 to m
  t(k) = a(k)*y(k)
  s(k+1) = t(k) + s(k)
end
c = s(m+1)
```

   For each iteration step we have

$$t_k = a_k y_k (1+\xi_k) \tag{2}$$

and

$$
\begin{aligned}
s_{k+1} &= (t_k + s_k)(1+\eta_k) \\
&= a_k y_k (1+\xi_k)(1+\eta_k) + s_k(1+\eta_k) \\
&= a_k y_k (1+\xi_k)(1+\eta_k) \\
&\quad + a_{k-1} y_{k-1}(1+\xi_{k-1})(1+\eta_{k-1})(1+\eta_k) \\
&\quad + s_{k-1}(1+\eta_{k-1})(1+\eta_k) \\
&= \sum_{h=1}^{k} a_h y_h (1+\xi_h) \prod_{l=1}^{h}(1+\eta_l) \\
&= \sum_{h=1}^{k} a_h y_h (1+\epsilon_{h+1})
\end{aligned} \tag{3}
$$

where $(1+\epsilon_{h+1}) = (1+\xi_h)\prod_{l=1}^{h}(1+\eta_l)$. If we assume that $|\eta_k|$ and $|\xi_k|$ are both bound by η (which is reasonable), that is, we have $\max\{|\eta_k|, |\xi_k|\} < \eta$, then

$$
\begin{aligned}
(1-\eta)^k \quad &\leq 1+\epsilon_k \leq (1+\eta)^k \\
(1-\eta)^k - 1 \leq &\quad\ \ \epsilon_k \leq (1+\eta)^k - 1
\end{aligned} \tag{4}
$$

whereby

$$
|\epsilon_k| < \max\left\{\left|(1-\eta)^k - 1\right|, \left|(1+\eta)^k - 1\right|\right\} = \left|(1+\eta)^k - 1\right|. \tag{5}
$$

Therefore, the numerical representation of the dot-product is

$$
\hat{c} = \mathbf{a}^H \mathbf{y}(1+\epsilon) \tag{6}
$$

where for ϵ it holds that

$$
|\epsilon| < |(1+\eta)^n - 1|. \tag{7}
$$

η is the relative accuracy of the corresponding numerical representation and n is the number of elements in vector $\mathbf{a}$.

Note that we cannot estimate the bounds of ϵ within the floating point arithmetic whose bounds it estimates, since the numerical accuracy is not sufficient. We therefore have to estimate these bounds with, for example, Taylor expansion.

2. Let $\mathbf{A}$ be an $N \times N$ hermitian matrix, $\mathbf{A} = \mathbf{A}^H$, $\mathbf{v}_i$ be the eigenvectors of $\mathbf{A}$, λ_i the corresponding eigenvalues and $|\lambda_i| \leq |\lambda_{i+1}|$.

Then $\mathbf{A}$ is unitary similar to a diagonal matrix, that is, we can write

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^H$$

where $\mathbf{Q}\mathbf{Q}^H = \mathbf{Q}^H\mathbf{Q} = \mathbf{I}$ and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_N \end{bmatrix}.$$

Then we can write

$$\begin{aligned} ||\mathbf{A}||_2^2 &= \max_{\mathbf{x}\neq 0} \frac{||\mathbf{A}\mathbf{x}||_2^2}{||\mathbf{x}||_2^2} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^H\mathbf{A}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^H\mathbf{A}\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \\ &= \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^H\mathbf{Q}\mathbf{D}\mathbf{Q}^H\mathbf{Q}\mathbf{D}\mathbf{Q}^H\mathbf{x}}{\mathbf{x}^H\mathbf{x}} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^H\mathbf{Q}\mathbf{D}\mathbf{D}\mathbf{Q}^H\mathbf{x}}{\mathbf{x}^H\mathbf{Q}\mathbf{Q}^H\mathbf{x}} \end{aligned}$$

since $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}$. Substituting $\mathbf{y} = \mathbf{Q}^H\mathbf{x}$ yields

$$||\mathbf{A}||_2^2 = \max_{\mathbf{y}\neq 0} \frac{\mathbf{y}^H\mathbf{D}^H\mathbf{D}\mathbf{y}}{\mathbf{y}^H\mathbf{y}} = \max_{\mathbf{y}\neq 0} \frac{||\mathbf{D}\mathbf{y}||_2^2}{||\mathbf{y}||_2^2} = ||\mathbf{D}||_2^2$$

Note that the largest element of $\mathbf{D}$ is λ_N and thus $||\mathbf{D}||_2^2 = \lambda_N$, whereby $||\mathbf{A}||_2^2 = \lambda_N$.

3. Eigenvalues λ_k of autocorrelation matrices are positive $\lambda_k > 0$, but numerical errors can make that limit approximate $\lambda_k \gtrapprox 0$. Addition of a sufficiently large constant μ will make the eigenvalues strictly positive $\lambda_k + \mu > 0$. For the autocorrelation matrix $\mathbf{R}$ this corresponds to $\mathbf{R} + \mu\mathbf{I}$, that is, regularisation of $\mathbf{R}$.

   In LMMSE estimation we invert matrix $\mathbf{R}_x + \sigma_v^2\mathbf{I}$, that is, we invert $\mathbf{R}_y = \mathbf{R}_x + \sigma_v^2$. Clearly this corresponds to regularisation of $\mathbf{R}_x$. However, $\mathbf{R}_x$ is generally estimated from $\mathbf{R}_y$ by $\mathbf{R}_x = \mathbf{R}_y - \sigma_v^2\mathbf{I}$, which is the opposite of regularisation and we actually make the conditioning worse. The larger σ_v^2 is, the more prominent the problem becomes. However, since we do not need to invert $\mathbf{R}_x$ that is not an immediate problem.

   On the other hand, if $\mathbf{R}_y$ is ill-conditioned from the beginning, we might benefit from regularisation. Here are three possible methods:

   (a) Originally we had

   $$\mathbf{H} = \mathbf{R}_x(\mathbf{R}_x + \sigma_v^2)^{-1} = (\mathbf{R}_y - \sigma_v^2\mathbf{I})\mathbf{R}_y^{-1} \tag{8}$$

but by replacing $\mathbf{R}_y$ by $\mathbf{R}_y + \mu\mathbf{I}$ we obtain

$$\hat{\mathbf{H}} = (\mathbf{R}_y + (\mu - \sigma_v^2)\mathbf{I})(\mathbf{R}_y + \mu\mathbf{I})^{-1}.$$

By choosing $\mu = \sigma_v^2$ we obtain a convenient form

$$\hat{\mathbf{H}} = \mathbf{R}_y(\mathbf{R}_y + \mu\mathbf{I})^{-1}.$$

(b) From Eq. 8 we see that we are inverting only $\mathbf{R}_y$ and we need to regularise only $\mathbf{R}_y$ and we can leave $\mathbf{R}_y - \sigma_v^2\mathbf{I}$ intact. Then

$$\hat{\mathbf{H}} = (\mathbf{R}_y - \sigma_v^2\mathbf{I})(\mathbf{R}_y + \mu\mathbf{I})^{-1}.$$

Heuristically, this has the benefit that we regularise (=corrupt) only those matrices that require regularisation.

(c) By rewriting we obtain

$$\mathbf{H} = (\mathbf{R}_y - \sigma_v^2\mathbf{I})\mathbf{R}_y^{-1} = \mathbf{I} - \sigma_v^2\mathbf{R}_y^{-1}$$

whereby, again, we need only regularise $\mathbf{R}_y$ and have

$$\hat{\mathbf{H}} = \mathbf{I} - \sigma_v^2(\mathbf{R}_y + \mu\mathbf{I})^{-1}$$

We observe that the choice of regularisation method is not trivial. Each of the above methods are well-warranted and we do not have general means of choosing a method. Our proposition is to analyse results in speech enhancement with both objective measures and listening tests in order to find the superior method.