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Plucked-String Models: From the Karplus-Strong Algorithm to Digital Waveguides and Beyond

The emergence of what is called *physical modeling* and *model-based sound synthesis* is closely related to the development of computational simulations of plucked string instruments. Historically, the first physical approaches (Hiller and Ruiz 1971a, 1971b; McIntyre and Woodhouse 1979; McIntyre, Schumacher, and Woodhouse 1983) were followed by the *Karplus-Strong* (KS) *algorithm* (Karplus and Strong 1983). The KS algorithm was discovered as a simple computational technique that seemingly had nothing to do with physics. Soon thereafter, Julius Smith and David Jaffe showed a deeper understanding of its relation to the physics of the plucked string (Smith 1983; Jaffe and Smith 1983).

Later, Julius Smith generalized the underlying ideas of the KS algorithm by introducing the theory of *digital waveguides* (Smith 1987). Digital waveguides are physically relevant abstractions yet computationally efficient models, not only for plucked strings, but for a variety of one-, two-, and three-dimensional acoustic systems (Van Duyne and Smith 1993; Savioja, Rinne, and Takala 1994; Van Duyne, Pierce, and Smith 1994). Further investigations embodied these ideas in more detailed synthesis principles and implementations, resulting in high-quality and realistic syntheses of plucked string instruments (Sullivan 1990; Karjalainen and Laine 1991; Smith 1993; Karjalainen, Välimäki, and Jánosy 1993; Välimäki, Huopaniemi, Karjalainen, and Jánosy 1996). A recent overview of research in this field is given by Smith (1996).

The equivalence of Karplus-Strong and digital waveguide formulations in sound synthesis was al–ready known when the waveguide theory appeared (Smith 1987, 1992, 1997); however, the relation has never been explicated in full detail. The first aim of this article is to show how the more "physical" waveguide model of a plucked string can be reduced to an extended form of the Karplus-Strong type that we call the *single delay-loop* (SDL) model. For a linear and time-invariant (LTI) case, this reduction is relatively straightforward, and results in a computationally more efficient digital filter structure. (Note that the historical order of the KS algorithm and digital waveguides is the reverse of their logical order, since the generalization was not developed until after the KS algorithms was designed. This article's title reflects the historical evolution: the "beyond" refers to recent generalizations and extensions of both concepts.)

The second aim of this article is to discuss further extensions to the basic SDL models that make them capable of simulating plucking styles, beats in string vibration, sympathetic vibrations, and resonant strings. Such techniques have already been proposed and studied (Jaffe and Smith 1983; Smith 1993; Karjalainen, Välimäki, and Jánosy 1993). Here we discuss them in the context of our recent implementations of plucked-string models.

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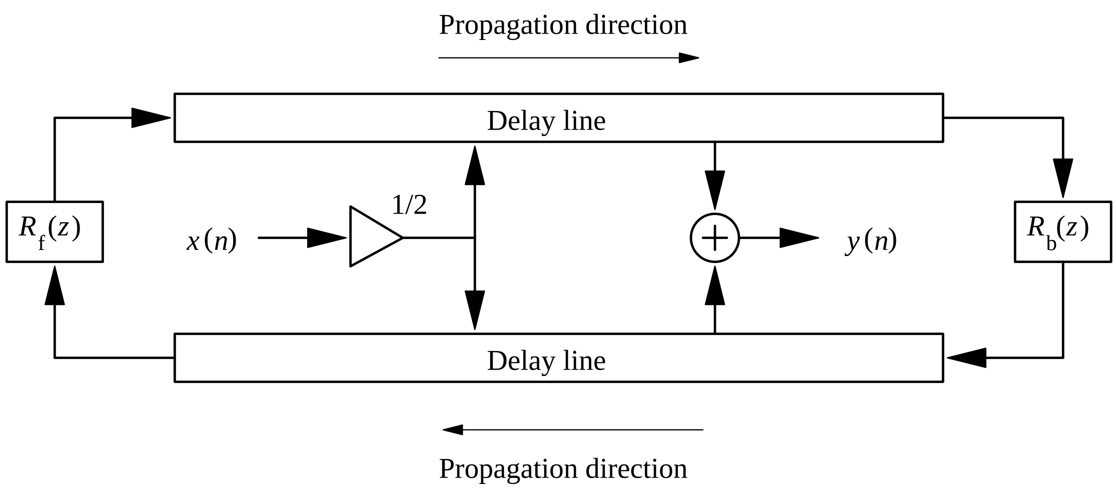
Fundamentals of String Behavior

The behavior of a vibrating string with a plucked excitation can be described in terms of two traveling waves traversing the string in opposite directions and reflecting back at the string terminations (Elmore and Heald 1969; Fletcher and Rossing 1991). When we assume that during autonomous vibration the string is an LTI system, we can model it as shown in Figure 1 (Smith 1987, 1992). The two delay lines

Figure 1. A bidirectional digital waveguide model for a terminated string.



can be interpreted as a digitized d'Alembert's solution to the one-dimensional lossless wave equation. The two waveforms travel through the delay lines and reflect at reflection filters $R_f(z)$ and $R_b(z)$ which produce phase inversion and slight frequency-dependent damping. The input signal $x(n)$ is summed into both delay lines just as output signal $y(n)$ is taken as a sum of the wave-variable values in the two delay lines at the observation point. This digital waveguide-modeling approach yields efficient implementations for real-time sound synthesis. For the digital waveguide of Figure 1, a further assumption is needed: All signals to be modeled must be bandlimited to below one-half of the sampling rate. Owing to the LTI assumption, string losses and dispersions can be commuted between any driving or observation points (Smith 1992, 1997). This allows the use of ideal delay lines that are computationally very efficient.

In the string model of Figure 1, the input and output signals can be of any wave-variable type, such as displacement, velocity, acceleration, or slope (Smith 1992; Morse 1976). An interesting case is to select acceleration as the wave variable, since then an ideal pluck corresponds to a unit impulse (Smith 1983; Karjalainen and Laine 1991).

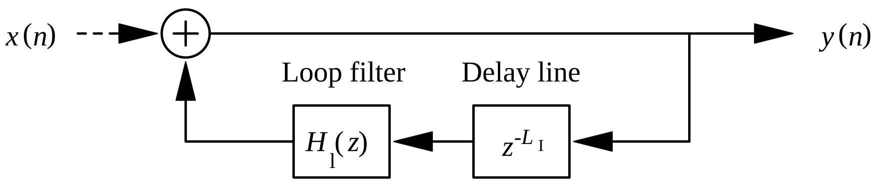
Further using the above simplification principles, it is possible to commute the elements of a terminated, dispersive, and lossy string into the form illustrated in Figure 2, provided that the output signal is taken to be a single traveling-wave component. In this extreme case, the losses and the dispersion are lumped at a single point in the round trip along the string: at the loop filter $H_1(z)$. When the loop filter is a two-point average $y(n) = [x(n) + x(n - 1)]/2$, and when the initial conditions (i.e., the initial contents of the delay line) that are used to pluck the string are taken to be random numbers, the well-known Karplus-Strong algorithm for plucked-string sounds is obtained (Karplus and Strong 1983; Jaffe and Smith 1983). Note that the original algorithm uses no explicit input signal.

From Bidirectional Digital Waveguides to SDL Models

Above, we discussed two computational models for stringed musical instruments that provide the basis for efficient real-time synthesis. The case of extreme simplicity and efficiency, the KS model of Figure 2, is certainly an oversimplification for anything but rudimentary synthesis. More detailed models are needed both for high-quality sound synthesis and for theoretical understanding in physical modeling. In this section, we derive the relations between the bidirectional digital waveguide and the SDL formulations in detail.

Let us consider the relation of the two basic formulations, the bidirectional digital waveguide model and the single delay-loop model. We will analyze two cases: a string with (1) a bridge output and (2) a pickup output. An excitation—such as a pluck—in a real physical string initiates wave components that travel independently in opposite directions. The output of the string—e.g., the force at

Figure 2. A simplified linear string model as a single delay-loop (SDL) structure. The Karplus-Strong algorithm is a special case in which the excitation is given as the initial state of the delay line.



the bridge of an acoustic instrument, or the pickup voltage in an electric guitar—reacts to both wave components. The effects of the excitation and pickup positions are easily simulated in the waveguide model that is based on a dual delay line, as depicted in Figure 1. However, for sound-synthesis purposes, the SDL realization, such as in Figure 2, is more efficient. Also, it is interesting from a theoretical point of view to formulate an SDL model that includes the effects of the excitation and pickup positions. It has been shown that an ideal acceleration or velocity input into a string model (corresponding to plucking or striking the string, respectively) can be approximated by a unit impulse (Smith 1992). Thus, by assuming linearity and time invariance, we can naturally think in terms of impulse responses, and interpret the string model as a linear filter.

Plucked String with Bridge Output

In the discussion to follow, we describe the transfer functions of the model components in the Laplace transform domain. The Laplace transform is an efficient tool in linear continuous-time systems theory. In particular, time-domain integration and derivative operations transform respectively into division and multiplication by the Laplace variable s. We can replace the complex variable s with $j\omega$ (where j is the imaginary unit $\sqrt{-1}$, ω is the radian frequency, which is equal to $2\pi f$, and f is the frequency in Hz) to derive the corresponding representation in the Fourier transform domain, i.e., the frequency domain. Later, we approximate the continuous-time system by a discrete-time system in the Z-transform domain. For more information on Laplace, Fourier, and Z-transforms, see a standard textbook on signal processing, such as that by Oppenheim, Willsky, and Young (1983).

In Figure 3, we redefine the dual delay-line waveguide model for an ideally plucked string with transversal bridge force as an output. This situation is applicable to the simulation of the acoustic guitar, for example. In our notation, $\mathcal{H}_{A,B}(s)$ refers to the transfer function from point A to point B. Note that we have divided the pluck excitation $\mathcal{X}(s)$ into two parts, $\mathcal{X}_1(s)$ and $\mathcal{X}_2(s)$, such that $\mathcal{X}_1(s) = \mathcal{X}_2(s) = \mathcal{X}(s)/2$.

We can first simplify the model by deriving an equivalent single excitation at point E1 that corresponds to the net effect of the two excitation components at points E1 and E2. When we assume that the bridge termination point (R1, R2) is to the right of the input point (E1, E2) as in Figure 1, the equivalent single excitation at E1 can be expressed as

$$\begin{aligned} \mathcal{X}_{E1,eq}(s) &= \mathcal{X}_1(s) + \mathcal{H}_{E2,L2}(s)\mathcal{R}_f(s)\mathcal{H}_{L1,E1}(s)\mathcal{X}_2(s) \\ &= ½[1 + \mathcal{H}_{E2,E1}(s)]\mathcal{X}(s) = \mathcal{H}_E(s)\mathcal{X}(s), \end{aligned} \tag{1}$$

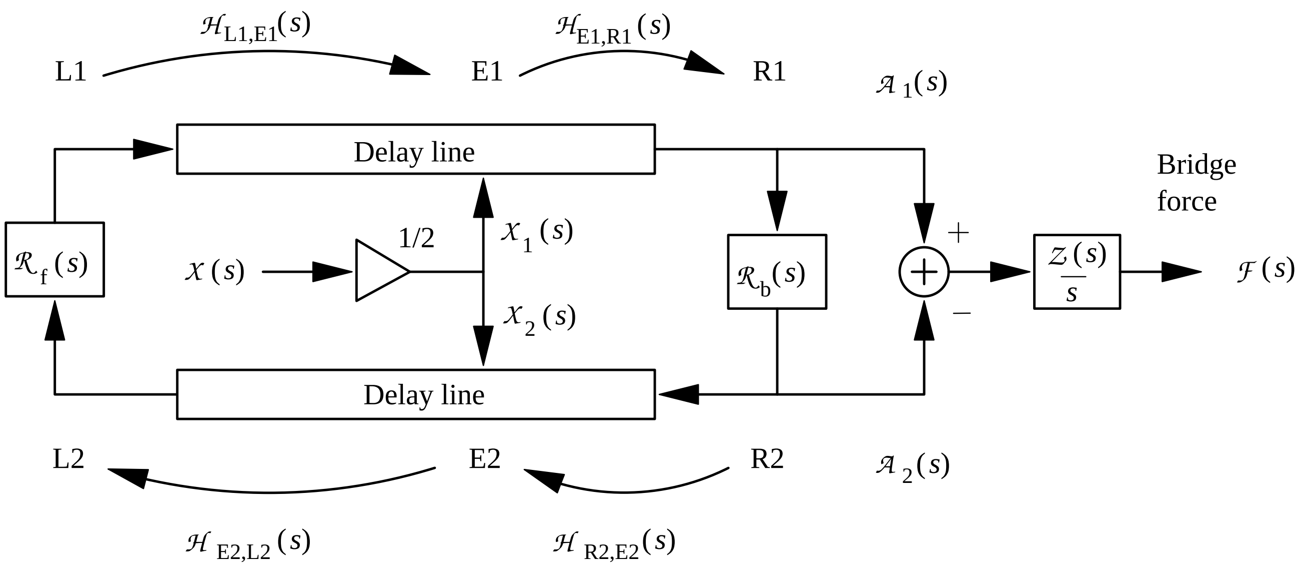
where subscript "eq" stands for "equivalent," and $\mathcal{X}_{E2,E1}(s)$ is the left-side transfer function from E2 to E1 consisting of the partial transfer functions from E2 to L2 and L1 to E1, and the reflection function $\mathcal{R}_f(s)$. Thus, $\mathcal{H}_E(s)$ is the equivalent excitation transfer function.

The output signal of interest is the transverse force $\mathcal{F}(s)$ at the bridge. It can be elaborated as

$$\begin{aligned} \mathcal{F}(s) &= \mathcal{F}_+(s) + \mathcal{F}_-(s) = \mathcal{Z}(s)[\mathcal{V}_+(s) - \mathcal{V}_-(s)] \\ &= \mathcal{Z}(s)[\mathcal{A}_1(s) - \mathcal{A}_2(s)]/s, \end{aligned} \tag{2}$$

where the "+" and "–" subscripts denote the two opposite propagation directions. Equation 2 states that the bridge force is the bridge impedance $\mathcal{Z}(s)$ times the difference of the string velocity components $\mathcal{V}_+(s)$ and $\mathcal{V}_-(s)$ at the bridge. The acceleration difference $\mathcal{A}_1(s) - \mathcal{A}_2(s)$ is integrated (operator $1/s$) to yield velocity difference $\mathcal{V}_+(s) - \mathcal{V}_-(s)$. Hence

Figure 3. Dual delay-line waveguide model for a plucked string with output at the bridge.



$$\begin{aligned}\mathcal{F}(s) &= \mathcal{Z}(s)[\mathcal{A}_1(s) - \mathcal{A}_2(s)]/s \\ &= \mathcal{Z}(s)[\mathcal{A}_1(s) - \mathcal{R}_b(s)\mathcal{A}_1(s)]/s \\ &= \mathcal{Z}(s)[1 - \mathcal{R}_b(s)]\mathcal{A}_1(s)/s \\ &= \mathcal{H}_B(s)\mathcal{A}_1(s),\end{aligned} \tag{3}$$

where $\mathcal{H}_B(s)$ is the acceleration-to-force transfer function at the bridge. Now

$$\mathcal{A}_1(s) = \mathcal{H}_{E1,R1}(s)\mathcal{X}_{E1,eq}(s) + \mathcal{H}_{loop}(s)\mathcal{A}_1(s), \tag{4}$$

where

$$\mathcal{H}_{loop}(s) = \mathcal{R}_b(s)\mathcal{H}_{R2,E2}(s)\mathcal{H}_{E2,E1}(s)\mathcal{H}_{E1,R1}(s), \tag{5}$$

i.e., $\mathcal{H}_{loop}(s)$ is the transfer function when the signal is circulated once around the loop. Thus, the sum terms of Equation 4 correspond respectively to the equivalent excitation signal $\mathcal{X}_{E1,eq}(s)$ transferred to point R1, and to the signal $\mathcal{A}_1(s)$ transferred once along the loop. This yields

$$\begin{aligned}\mathcal{A}_1(s) &= \mathcal{H}_{E1,R1}(s)\frac{1}{1 - \mathcal{H}_{loop}(s)}\mathcal{X}_{E1,eq}(s) \\ &= \mathcal{H}_{E1,R1}(s)\mathcal{S}(s)\mathcal{X}_{E1,eq}(s),\end{aligned} \tag{6}$$

where $\mathcal{S}(s)$ is the string transfer function that represents the recursion around the string loop. Putting all this together, we can solve for the overall transfer function from excitation to bridge output as

$$\begin{aligned}\mathcal{H}_{E,B}(s) &= \frac{\mathcal{F}(s)}{\mathcal{X}(s)} \\ &= \frac{1}{2}\left[1 + \mathcal{H}_{E2,R1}(s)\right]\frac{\mathcal{H}_{E1,R1}(s)}{1 - \mathcal{H}_{loop}(s)}\mathcal{Z}(s)\frac{1}{s}\left[1 - \mathcal{R}_b(s)\right],\end{aligned} \tag{7}$$

or more compactly, based on the above notation,

$$\mathcal{H}_{E,B}(s) = \mathcal{H}_E(s)\mathcal{H}_{E1,R1}(s)\mathcal{S}(s)\mathcal{H}_B(s), \tag{8}$$

which represents the cascaded contribution of each part in the physical string system.

At this point, we approximate the continuous-time model of the guitar in the Laplace transform domain with a discrete-time model in the Z-transform domain. This approximation is needed to make the model realizable in discrete-time form. Rewriting Equation 8 in the Z-transform domain, we obtain

$$H_{E,B}(z) = H_E(z)H_{E1,R1}(z)S(z)H_B(z), \tag{9a}$$

where

$$H_E(z) = \frac{1}{2}\left[1 + H_{E2,E1}(z)\right], \tag{9b}$$

and

$$S(z) = \frac{1}{1 - H_{\text{loop}}(z)}, \qquad (9c)$$

and

$$H_{\text{B}}(z) = Z(z)I(z)\left[1 - R_{\text{b}}(z)\right]. \qquad (9d)$$

Filter $I(z)$ is a discrete-time approximation of the time-domain integration operation. We interpret the result of Equation 9a by depicting a block diagram in Figure 4. It shows qualitatively the delays and the discrete-time approximations of the filter components of Equation 9 inherent in each part of the transfer function. Note that the top block in Figure 4 contains the minus sign of the reflection function included in $H_{\text{E2,E1}}(z)$. However, in the bottom block, the minus sign included in $R_{\text{b}}(z)$ cancels the minus sign of Equation 9d.

For practical sound synthesis, the model of Equation 9 and Figure 4 can be approximated by a simplified model depicted in Figure 5, without compromising the sound quality. The following simplifications have been made in Figure 5. The transfer function $H_{\text{E2,E1}}(z)$ in Equation 9b is almost a lossless delay, and we can drop the low-pass filter block in the excitation-position filter (or replace it with a constant slightly less than 1). Next, we notice that in normal playing conditions the wave propagation from the excitation point (E1) to the bridge position (R1) is also a nearly lossless delay, so that it can be left out without perceivable effects. The string loop $S(z)$ in Equation 9c cannot be reduced, because the delay and the low-pass-type loop filter are critical to the sound quality. Finally, the term $[1 - R_{\text{b}}(z)]$ in the bridge block $H_{\text{B}}(z)$ in Equation 9d can be approximated by the constant 2, since $R_{\text{b}}(e^{j\omega}) \approx -1$. The errors due to the reductions above can be compensated for with the timbre-control filter in Figure 5.

The sound-synthesis model of Figure 5 includes several controllable elements. The excitation table can contain any useful signals, such as a unit impulse or a complex aggregate excitation (Smith 1993; Karjalainen, Välimäki, and Jánosy 1993). For example, this is an efficient way to implement the resonances of the instrument body, since the body response can be commuted back to the excitation, owing to the LTI assumption. The table can also be a set of excitations to choose from or to interpolate between. The gain control is a simple multiplier. The timbre control may be a first- or second-order recursive filter that can be adjusted to attenuate or boost high frequencies, for a softer or sharper attack, respectively (Jaffe and Smith 1983). Note that the timbre-control filter must change with fundamental frequency to give a fixed percept of attack sharpness. The pluck-position control is an adjustable comb filter, where the delay corresponds to the time it takes for the excitation to travel the left-side route around from E2 to E1 in Figure 3. In practice, one can instead use the right-side delay from point E1 to E2 (which is normally shorter), since it will only slightly change the attack part of the model's impulse response. The comb filtering will create a series of zeros in the transfer function at frequencies $f_m = m/t_{\text{D}}$, where t_{D} is the delay (in seconds) of the comb filter, and m is an integer index ($m = 0, 1, 2, ...$).

The bridge-output integrator can be approximated by a first-order recursive low-pass filter whose cutoff frequency lies below the lowest fundamental frequency to be synthesized. One can also consider the pluck-position filter and the output integrator together, since at zero frequency the zero in the former filter will be canceled by a pole in the latter. Figure 6 illustrates the effect of both filters as found in their combined magnitude transfer function. The plucking position is expressed as a distance to the bridge relative to the string length, e.g., 50 percent refers to the middle point of the string (see Figure 6a). This effect is equivalent to the traditional interpretation of plucked-string behavior (see Fletcher and Rossing, Figure 2.5 [page 39] and Figure 2.7 [page 41]).

The string loop in Figure 5 is similar to the one in Figure 2. The delay block must allow for fine tuning of the time delay to achieve all desired pitch values (Jaffe and Smith 1983). This is accomplished by fractional delay filtering techniques, such as first-order all-pass filtering or Lagrange interpolation (Laakso, Välimäki, Karjalainen, and Laine 1996). Also, the loop filter has to be controllable, so that the decay of the harmonic components can be adjusted properly according to the string length and other varying parameters of the string. One popular choice is a one-pole digital loop filter with two parameters: a DC gain and a cutoff frequency parameter (Jaffe and Smith 1983; Välimäki et al. 1996).

Figure 4. A diagram characterizing the building blocks of the plucked string with force output at the bridge.

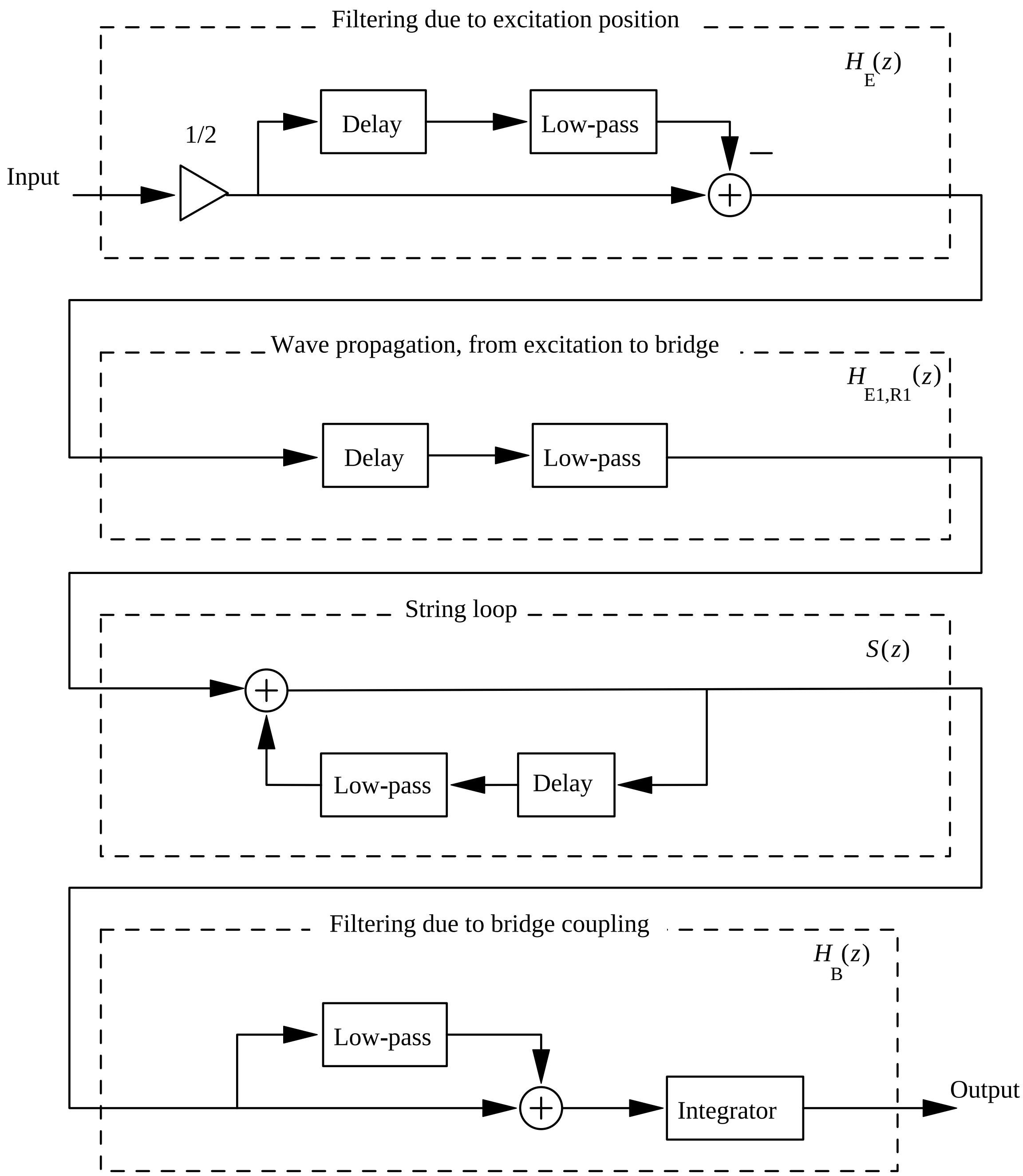


Figure 5. A schematic sound-synthesis model for plucked string instruments. The delay in the string loop must be continuously variable, which can be achieved with a fractional delay filter.

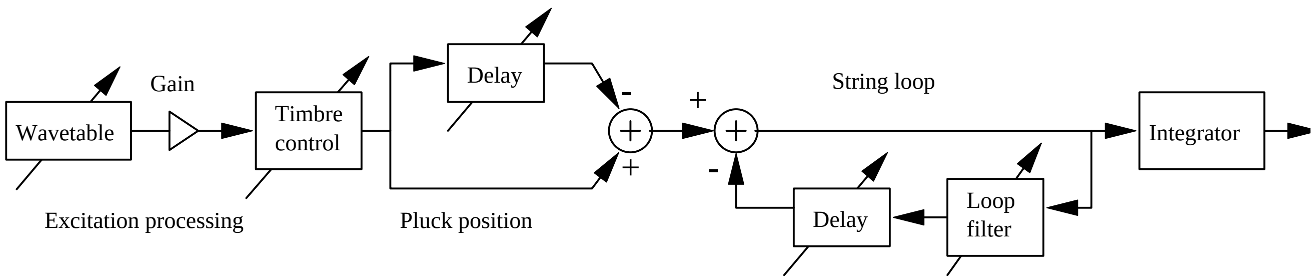
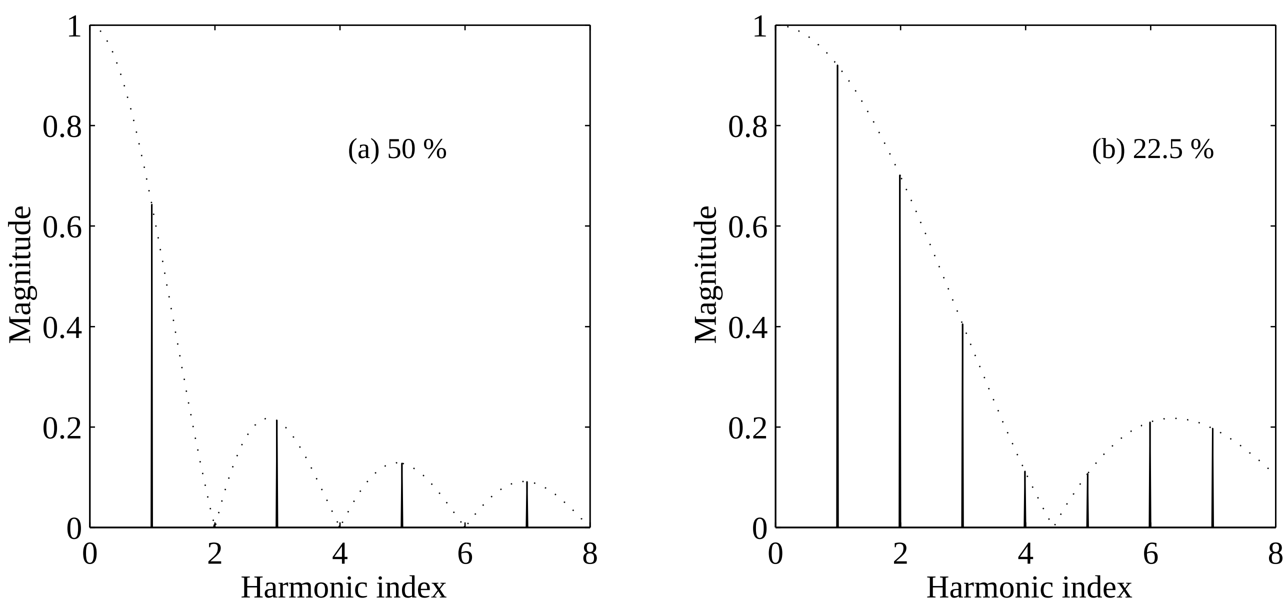
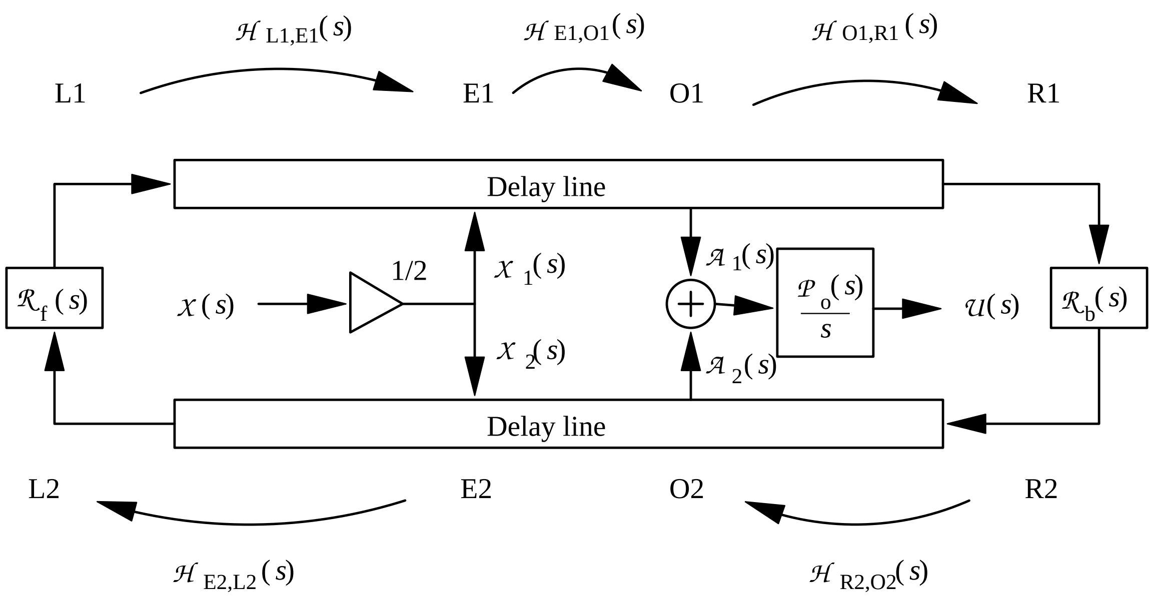


Figure 6. Two examples of the comb-filter effect caused by excitation-position and bridge-integrator filtering, when the relative distance from the plucking point to the bridge is 50 percent (a) and 22.5 percent (b) of the string length.



(a) (b)

Figure 7. Dual delay-line waveguide model for a plucked string with a pickup output.



Plucked String with Pickup Output

In the next analysis, we derive the transfer function and the synthesis block diagram for a plucked string when the output is taken by a velocity-sensitive sensor, such as a magnetic pickup in an electric guitar. Figure 7 shows a dual delay-line waveguide model for this case. The pickup position is denoted by (O1, O2).

A comparison of Figures 3 and 7 reveals that the excitation transfer function is the same in both cases. This implies that Equation 4 is valid also for a string with a pickup output, provided that the pickup point (O1, O2) is to the right of the excitation point (E1, E2).

The output signal is now a variable, such as voltage from a magnetic pickup, that is proportional to the transversal string velocity (sum of the left- and right-going components) at the pickup position. Velocity $\mathcal{V}(s)$ is proportional to the time integral of acceleration, and the voltage output from the pickup $\mathcal{U}(s)$ can be expressed as

$$\begin{aligned}\mathcal{U}(s) &= \mathcal{P}_o(s)\mathcal{V}(s)\\ &= \frac{\mathcal{P}_o(s)}{s}\left[\mathcal{A}_1(s)+\mathcal{A}_2(s)\right]\\ &= \frac{\mathcal{P}_o(s)}{s}\left[\mathcal{A}_1(s)+\mathcal{H}_{O1,R1}(s)\mathcal{R}_b(s)\mathcal{H}_{O2,E2}(s)\mathcal{A}_1(s)\right] \qquad (10)\\ &= \frac{\mathcal{P}_o(s)}{s}\left[1+\mathcal{H}_{O1,O2}(s)\right]\mathcal{A}_1(s)\\ &= \mathcal{H}_P(s)\mathcal{A}_1(s),\end{aligned}$$

where the pickup transfer function $\mathcal{P}_o(s)$ models the effect of the pickup microphone on guitar sound. For a magnetic pickup, $\mathcal{P}_o(s)$ is typically a second-order low-pass filter. Notice that it is combined with the integrator block $1/s$ at the output in Figure 7.

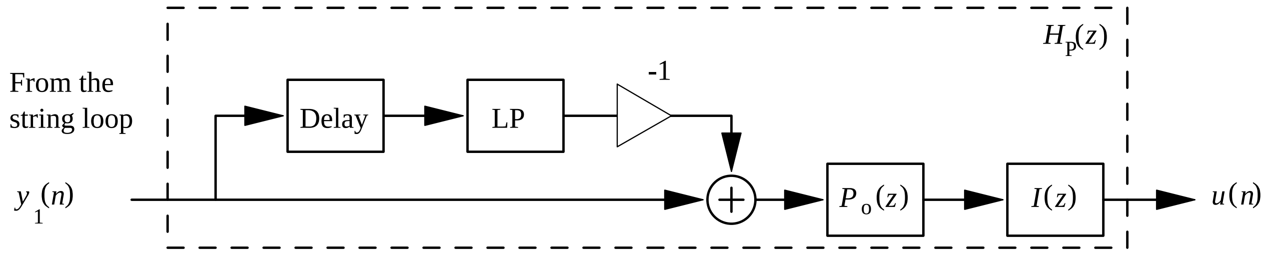
Corresponding to Equation 4, we can now write

$$\mathcal{A}_1(s) = \mathcal{H}_{E1,O1}(s)X_{E1,eq}(s)+\mathcal{H}_{loop}(s)\mathcal{A}_1(s), \qquad (11)$$

which yields

$$\begin{aligned}\mathcal{A}_1(s) &= \mathcal{H}_{E1,O1}(s)\frac{1}{1-\mathcal{H}_{loop}(s)}X_{E1,eq}(s)\\ &= \mathcal{H}_{E1,O1}(s)\mathcal{S}(s)X_{E1,eq}(s).\end{aligned} \qquad (12)$$

Figure 8. A pickup output block for the string model of Figure 7. The block LP is a low-pass filter.



Based on Equations 1, 10, and 12, we can solve for the overall excitation-to-pickup transfer function,

$$\mathcal{H}_{E,P}(s) = \frac{\mathcal{U}(s)}{\mathcal{X}(s)} = \frac{1}{2}\left[1 - \mathcal{H}_{E2,E1}(s)\right]\frac{\mathcal{H}_{E1,O1}(s)}{1 - \mathcal{H}_{loop}(s)}\frac{\mathcal{P}_o(s)}{s}\left[1 + \mathcal{H}_{O1,O2}(s)\right], \quad (13)$$

or more compactly, based on the above notations,

$$\mathcal{H}_{E,P}(s) = \mathcal{H}_E(s)\mathcal{H}_{E1,O1}(s)\mathcal{S}(s)\mathcal{H}_P(s)\,. \quad (14)$$

If the pickup point O is to the left of the excitation point (E1), we can derive a similar formulation where the indices 1 and 2 are interchanged, as are the string terminations, L and R.

As with the case of the bridge output, we now approximate the continuous-time transfer function in the Laplace transform domain with a discrete-time transfer function in the Z-domain:

$$H_{E,P}(z) = H_E(z)H_{E1,O1}(z)S(z)H_P(z)\,. \quad (15)$$

The differences between the bridge-output model (Equation 7) and the pickup-output model (Equation 13) are twofold. First, the wave propagation from the excitation point to the output, $\mathcal{H}_{E1,R1}(s)$ versus $\mathcal{H}_{E1,O1}(s)$, covers different distances along the string. Due to very low losses of wave propagation during normal playing conditions, this difference is negligible. Second, the transfer functions related to the output couplings are very different. While the difference of accelerations at the bridge R has a relatively flat response (Equation 3 and Figure 3), the summation of the two acceleration waves at the pickup point O creates a comb-filter effect (Equation 10 and Figure 7) similar to the excitation point filtering. Thus, a string-instrument model with a string-velocity pickup has two cascaded comb filters to color the response, instead of just the one in Figure 4. A block diagram of the string-output stage for a pickup is shown in Figure 8. Note that again the minus sign due to the reflection from the end of the string is explicitly shown with multiplication by -1 (compare with Figure 4).

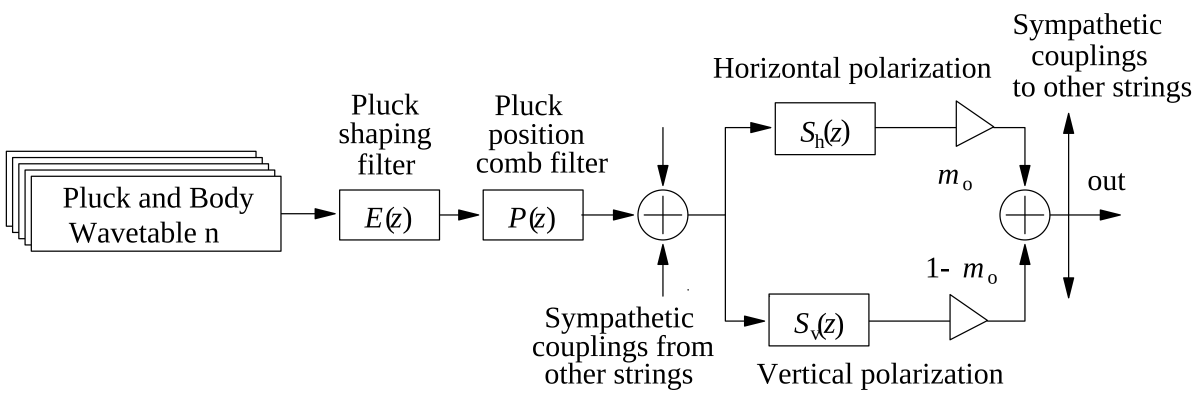
A sound-synthesis model for a plucked string with a pickup output will be the same as the diagram shown in Figure 5, except that a pickup-position comb filter and a pickup filter, $\mathcal{P}_o(s)$, must be cascaded. The comb filter can be similar to the pluck-position filter of Figure 5, and the pickup filter can be a second-order low-pass filter.

The synthesis models derived above can be further simplified using the commuted aggregate excitation techniques (Smith 1993; Karjalainen, Välimäki, and Jánosy 1993), whereby the impulse responses of other parts, such as the body, are preconvolved with the primary excitation and stored in a wavetable. If all the components apart from the string loop are aggregated, we approach a variant of the basic KS algorithm that has an explicit input signal. All variations are possible, from a fully aggregated excitation containing even the string loop (that is, pure wavetable synthesis), to the SDL model of Figure 5, or to the full bidirectional waveguide model of Figure 3. This shows the compatibility of physical modeling with more traditional synthesis techniques, such as source-filter modeling or sampling. The more simplified versions are computationally more efficient, naturally, but the price is paid with decreased flexibility of parametric control.

More Extensions to the SDL Models

The SDL models derived above are flexible building blocks when one is developing model-based sound synthesis of plucked string instruments. The SDL

Figure 9. Single delay-loop structure with consolidated pluck and body wavetables, pluck-position comb filter, dual-polarity string models, and sympathetic couplings between strings (Välimäki et al. 1996).



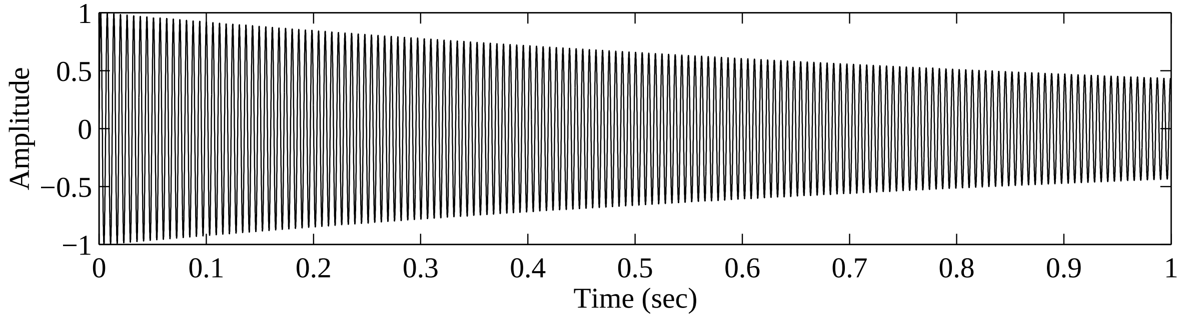
models show improved efficiency compared to the bidirectional digital waveguides, because the basic DSP building blocks are maximally consolidated and simplified. For example, a single fractional delay filter for fine-tuning the pitch is sufficient. However, if the string behavior contains essential nonlinearities or time-varying characteristics, bidirectional waveguide formulations are needed (see, for example, Karjalainen, Backman, and Pölkki 1993).

It is possible to add further details to the SDL models to improve the naturalness and other sound-quality features. Figure 9 illustrates an extended model in which additional properties are implemented (Välimäki et al. 1996). The string model's excitation is realized with wavetables that store consolidated pluck excitations and body responses, for easy and extremely efficient (but non-physical) modeling of the body (Smith 1993; Karjalainen, Välimäki, and Jánosy 1993). Alternative wavetables can be applied for different pluck styles and qualities. A pluck-shaping filter $E(z)$ can be used to fine-tune the timbre of a single pluck-table excitation. The comb-notch filter effect caused by pluck position and pickup position are easy to add. The pluck-position filter $P(z)$ is shown in Figure 9, but the pickup-position filter is not included.

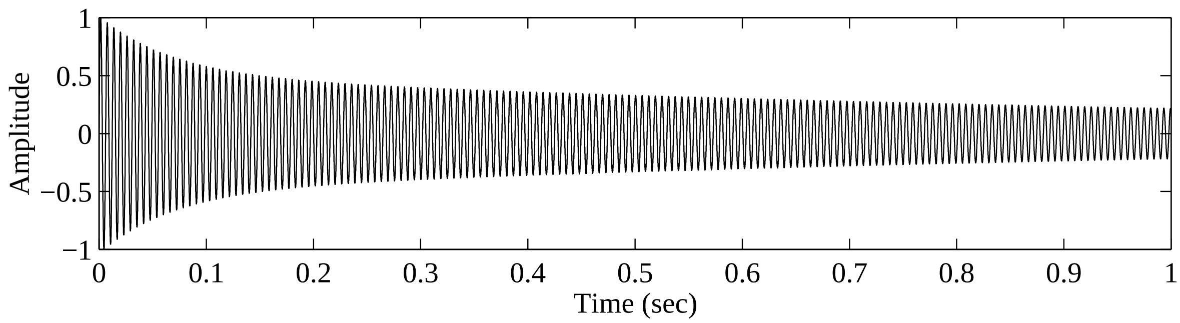
The beat effects caused by dual polarization of string vibration (horizontal and vertical with respect to the top plate) can be realized by mixing the outputs of two string models (Jaffe and Smith 1983). When the two models are slightly mistuned, a natural sounding beat effect results that reduces the "synthesizer-like" character. An example of the effect of mistuning the two polarization models is shown in Figure 10. In Figure10a, the model parameters are equal, and exponential decay is resulted. In Figure10b, the fundamental frequencies of the models are equal, but the loop-filter parameters are different, and a two-stage decay is produced. In Figure10c, the loop-filter parameters are equal, but the frequencies are mistuned to obtain a beating effect.

Several principles to simulate the sympathetic coupling between strings have been proposed (Jaffe and Smith 1983; Smith 1993; Välimäki et al. 1996). A physically correct method is a bridge-coupling filter presented by Smith (1993). A simple feedback coupling is added in Figure 9 to simulate the sympathetic coupling between strings. This approach is potentially unstable, because there is a feedback from the output of all strings to their inputs. When the feedback signals are attenuated using small gain coefficients between all outputs and inputs, it is possible to reach a stable simulation. Nevertheless, it would be safer to use a sympathetic coupling model that is inherently stable. Jaffe and Smith (1983) proposed using a separate bank of sympathetic strings that get their input signal from the output of the main strings that are plucked. This approach is always stable. However, it would be desirable to use the existing strings of the instrument model for generating sympathetic vibrations, and not implement separate models. In the following section, we introduce such a configuration.

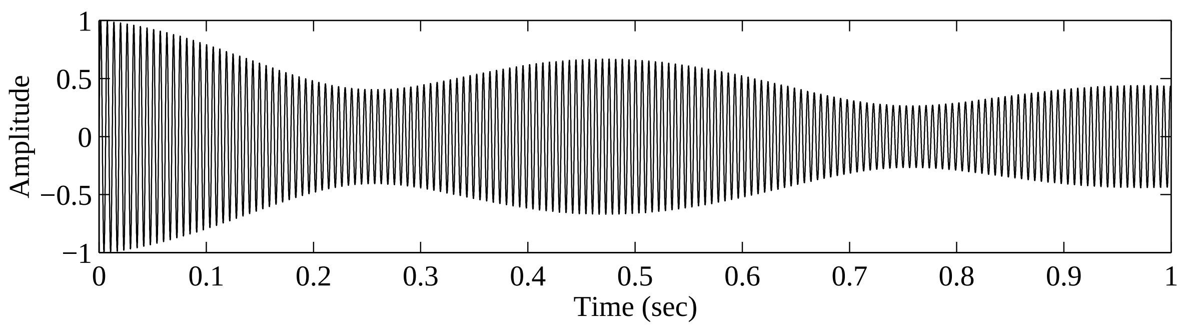
Figure 10. An example of the effect of mistuning the polarization models: equal parameter values (a), mistuned decay rates (b), and mistuned fundamental frequencies (c).



(a)

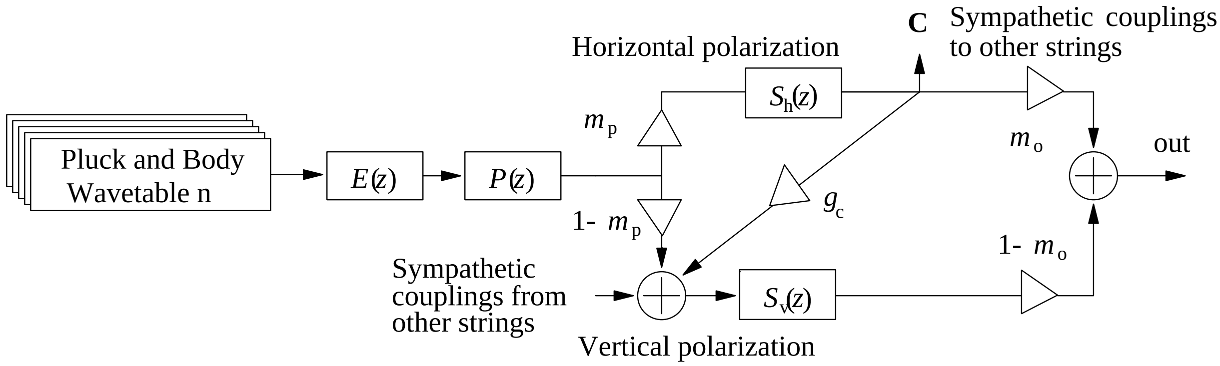


(b)



(c)

Figure 11. A modified plucked-string synthesis model with a new method to include sympathetic coupling between strings.



Coupling Phenomenon

Figure 11 illustrates a modified version of the plucked-string synthesizer for a single dual-polarization string. This time, the excitation for sympathetic vibrations is taken from one of the parallel strings that model the two polarizations (in Figure 11, the horizontal one is used). To avoid feedback, the input from other strings is added to the input of only those parallel strings that do not have sympathetic coupling output. This implies that input must be fed into the vertical string model in Figure 11, because the output is taken from the horizontal model. This kind of signal coupling is unconditionally stable, and produces realistic sympathetic coupling phenomena. Furthermore, separate sympathetic string models need not be implemented, since sympathetic vibrations are now produced in a natural manner by all the strings included in the synthesis model.

In the general form of this algorithm, there is a matrix **C** of coupling coefficients that determine the proportion of the output signal to be sent to a particular parallel string. This matrix can be written as

$$\mathbf{C} = \begin{bmatrix} g_{c1} & c_{12} & c_{13} & \cdots & c_{1N} \\ c_{21} & g_{c2} & c_{23} & & \\ c_{31} & c_{32} & g_{c3} & & \vdots \\ \vdots & & & \ddots & \\ c_{N1} & & \cdots & & g_{cN} \end{bmatrix} \qquad (16)$$

where N is the number of dual-polarization strings, the coefficients g_{ck} (for $k = 1, 2, 3, ..., N$) denote the gains of the output signal to be sent from the kth horizontal string to its parallel vertical string, and coefficients c_{mk} are the gains of the kth horizontal string output to be sent to the mth vertical string. There is a physical motivation to use real numbers less than 1 for all the elements of matrix **C**. However, the structure's stability does not depend on these values, since there is no feedback.

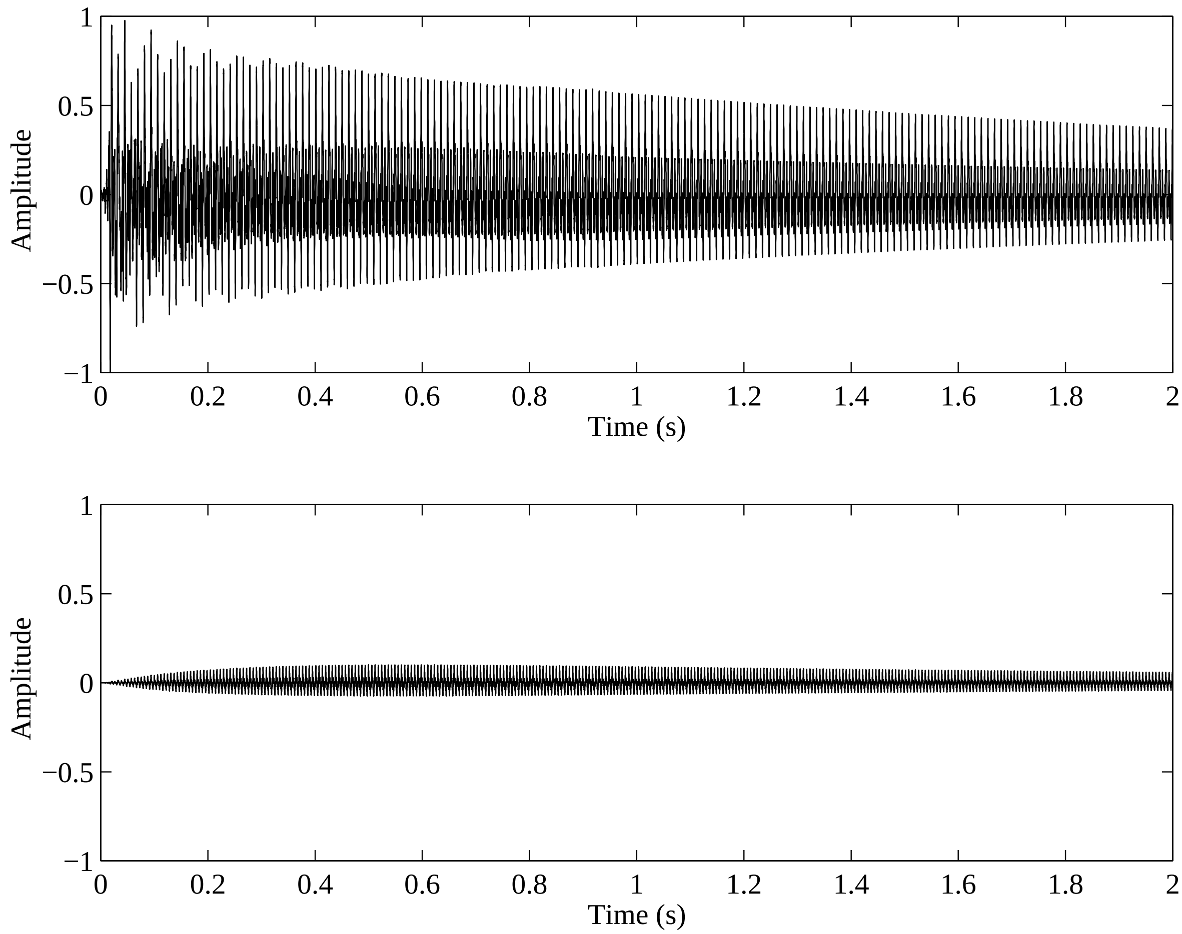
The model shown in Figure 11 also divides the excitation signal that is sent to the two polarizations, by using a mixing coefficient m_p that is chosen to have a value between 0 and 1. A non-zero value of parameter g_{ck} enables coupling of the two polarizations. If at the same time $m_p = 1$, the model for the vertical polarization becomes a resonance string that receives input only from the upper string model in Figure 11.

An example of sympathetic coupling is pictured in Figure 12. The primary vibration (the waveform displayed in the upper part of Figure 12) excites another string whose output signal is shown in the lower part of Figure 12. In this example, the fundamental frequency of the secondary string is an octave higher than that of the primary string. Notice the slow attack of the vibration in the coupled string (the lower part of Figure 12).

Still More Extensions, and Future Directions

Topics for future work in physical modeling of plucked strings includes pluck simulation, different kinds of nonlinearities, the interaction between the string and the body, modeling of the body response

Figure 12. An example of the simulation of sympa–thetic coupling using the model of Figure 11. The primary vibration (top) excites vibration in another string (bottom).



using a digital filter, and calibration of the model parameters. Some of these problems have been tackled in recent literature. Rank and Kubin (1997) proposed a nonlinear model for cases where the amplitude of string vibration is limited by contact with frets, such as in slap-bass playing techniques. A passive nonlinear filter structure was devised by Pierce and Van Duyne (1997). They presented an example where the nonlinear generation of missing harmonics in string vibrations was successfully simulated by their simple digital model. Digital filter approximations of the body response were discussed by Karjalainen and Smith (1996). In the computationally efficient synthesis models, it has been advantageous to avoid the use of a body model and instead use the principle of commuted waveguide synthesis. However, if one wishes to simulate the two-way interaction between the strings and the body, an explicit model for the body can be developed, although this is not required in the LTI case.

Calibration of the parameter values of a plucked-string model was tackled by Välimäki et al. (1996). The proposed technique was based on short-time

Fourier analysis of recorded plucked string tones. The excitation signal was obtained by inverse-filtering a recorded sound using the inverse transfer function of the string model. This enabled high-quality resynthesis of tones that were well behaved in that their harmonics decayed nearly exponentially. Problematic cases are tones where beats occur in harmonics or the decay rate of harmonics changes over time in some other way. A recent improvement to this problem is a method where the excitation signal is obtained by subtracting a sinusoidal model of the harmonics from an analyzed tone (Tolonen and Välimäki 1997; Välimäki and Tolonen 1998; Tolonen 1998). This method accounts for the time-varying decay rate of harmonics, and yields a clean excitation signal. A remaining problem in parameter calibration is to reliably estimate the pluck position from a recording of a plucked-string tone (Välimäki et al. 1996). It would be useful to extract this information and cancel its effect in the excitation signal. Thereafter, the excitation position would be a free parameter in the synthesis stage.

The realism of plucked-string synthesis could be further improved by including different kinds of side effects, such as those generated when the player slides his or her finger along a string (friction noise) or touches the body of a guitar (Jánosy, Karjalainen, and Välimäki 1994). Such sound effects could be incorporated by using samples that are triggered at the right time according to certain rules or, to obtain more expression and variation, by developing physical models for these effects. Furthermore, a virtual plucked string instrument should allow the player to use the complete range of playing styles, including the left-hand techniques such as pull-offs, hammer-ons, trills, slurs, and the use of a slide, as well as right-hand techniques such as fret-tapping. Jaffe and Smith (1983) discussed the simulation of some of these styles. Nevertheless, realistic and computationally efficient physical modeling of these playing techniques remains a future challenge.

Conclusions

In this article, we focused on algorithms for the synthesis of plucked-string tones. The relation and equivalence of the bidirectional digital waveguide and single delay-loop (SDL) models of plucked string instruments were discussed. The derivation of SDL models based on the digital waveguide approach was given for the cases of a bridge output and a pickup output. A further extension of the SDL model with several additional features for improved sound quality was presented, including a new way to simulate sympathetic vibrations.

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